Distributed Tracking of Network Size, Diameter, Radius, and Node Eccentricities in Open Multi-Agent Systems

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***Abstract*— We present a distributed algorithm enabling the dynamic tracking of diameter and radius (DR) of time-varying open networks (ON) together with nodes' eccentricities and network size (ES), called DR-ON-ES, which does not require the disclosure of nodes' identity nor any a priori information on the network. The convergence properties of DR-ON-ES are discussed within the framework of open multi-agent systems (OMASs): sufficient conditions are provided to ensure that the TPI of an OMAS executing DR-ON-ES is globally asymptotically open stable, meaning that all trajectories converge to within a neighborhood of the sought parameters. The parameters' estimations at equilibrium points are characterized in terms of mean expected value and mean squared error. As an ancillary result, we remark that DR-ON-ES exploits the OSTDMC Protocol – a novel distributed protocol formalized and analyzed in this manuscript – to achieve consensus on the time-varying maximum value of a set of signals fed locally to the agents of the network.**

***Index Terms*— Open Networks, Distributed Estimation, Dynamic Consensus, Dynamic Tracking, Graph.**

I. Introduction

The behavior of a large group of entities, such as robot teams, networks of computing units, sensor networks, smart grids, etc., can be captured by using the multi-agent system (MAS) paradigm. The interactions among the agents, influenced by sensing, communication, or physical coupling, are represented by a graph reflecting the network structure, where the nodes represent the agents and edges connecting the nodes represent these interactions. While most of the existing literature is limited to fixed-size networks, thus assuming that no agent may join or leave the network as time goes by, this article delves into the realm of open multi-agent systems (OMASs), where the number of agents within the network is time-varying. This characteristic is prevalent in all real engineering applications like the Internet of Things, vehicle platooning in multi-robot systems [1], [2], energy management in smart power grids [3], [4], online optimization in machine learning [5]–[8], consensus in cooperative networks [9], [10], and so on.

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The main contribution of this manuscript is to provide a distributed solution for tracking key graph parameters – specifically, the diameter, the radius, the nodes' eccentricity and their number – in open and anonymous networks, where the agents are free to join and leave as time goes by and the topology continuously evolves. The proposed algorithm is designed to handle such variability, making it well-suited for practical applications in mobile ad-hoc networks [11] where pairing the initial number of nodes (that might be known at the time of deployment) and consequent estimates can be used to detect potential disconnections; in multi-robot networks [12] where estimations on graphs' parameters may be instrumental for coordination and collision avoidance; sensor networks [13] where the knowledge of nodes and number of edges can be used as proxy metric for power consumption; in Internet of Things (IoT) applications [14] where the knowledge of the number of devices may be used to adjust the communication protocols so that everything runs smoothly. Moreover, the proposed algorithm offers the advantage of preserving the nodes' anonymity by non-disclosing of their identities. In dynamic networked control scenarios, explicitly assigning and revealing unique node identifiers can lead to several issues, including impractical operational constraints, ncreased vulnerability to targeted attacks, and privacy concerns. For instance, in smart grids, sensor data is often aggregated anonymously to prevent consumer profiling [15], while in vehicular networks, anonymous communication protocols can mitigate risks associated with tracking and spoofing [16].

A. Literature review and main contributions

1) Distributed estimation of graph parameters: Within our control community, various methods have been proposed in the literature for the distributed estimation of graph parameters in networks with a fixed number of agents. A brief collection of works that exemplify various employed strategies – such as consensus, leader election, statistical inference, random walks, random sampling – and enclosing the researchers most devoted to the topic, is given by [17]–[23] and reference therein. The main drawbacks of the algorithms proposed in these works are: the dependence on precise initial conditions to correctly estimate the parameter of interest, common to all but [21] that, instead, requires the centralized selection of one (or more) leader; the assumption of the existence of a unique identifier for each node which is known to all other nodes in the network [18], [20]; the partial knowledge of the network, such as bounds on the number of nodes [17], [20] or the diameter [17], [22].

The novelty of our research to this literature is the presentation of a fully distributed algorithm for the estimation and dynamic tracking of networks' Diameter and Radius (DR) of time-varying Open Networks (ON) with nodes' Eccentricities and Network Size (ES). The proposed algorithm, called DR-ON-ES, does not have any of the above-mentioned weak points, being resilient against re-initialization, avoiding the disclosure of nodes' identity, keeping the privacy of the local reference signal, and not requiring any a priori information of the network. We prove stability of an OMAS executing DR-ON-ES, showing its finite-time convergence and characterizing the estimations in terms of mean expected value and mean squared error. Recently, the open scenario was numerically tested in [24] – where the authors propose an exact algorithm to count the number of nodes and edges in the specific scenario of leader-follower networks – and in [25] – where the authors recast the size estimation problem into a distributed optimization problem. However. these algorithms lack of a theoretical characterization for open networks.

2) Open Multi-Agent Systems (OMASs): The concept of open systems has only recently gained attention in our control community for specific applications such as resource allocation [26], mode computation [27], optimization [28], [29], and learning [30]. This is mostly due to various conceptual difficulties in adapting control-theoretic notions such as state and stability when the size of the systems changes over time. Indeed, simple concepts like the distance between two states become unfeasible when the two states have different dimensions. Different strategies and perspectives have been considered to overcome these issues, such as embedding the time-varying set of agents in a time-invariant superset while considering constant the state of the agents that are not currently active [31]; exploiting gossip-based interactions [32]–[34] or leveraging time-scale separation principle between the rate at which agents join/leave the network and the rate of the protocol steps execution [9], [35] while achieving consensus on specific metrics, such as the average, the median, and the maximum, where the influence of additive nose has been analyzed in [36]; formulating graphon models in infinite dimensional spaces to represent arbitrary-size networks of linear dynamical systems [37]–[39].

The novelty of our research to this literature, is that of formulating general stability criteria for the class of *slowly expansive* and *paracontractive* OMASs as defined in our preliminary work [40]. We follow the general framework presented in [41], where proper definitions of state evolution, equilibria, and stability are established for discrete-time OMASs, together with stability criteria for the special class of *contractive* OMASs. Differently from [41], the difficulty in evaluating the distance between vectors in different spaces $\mathcal{S}_1 \neq \mathcal{S}_2$ is overcome by using the infinity norm instead of the Euclidean norm [42]–[46] (avoiding the need of normalizing the distances by the number of components) and by considering only the components in the intersection of the two spaces $\mathcal{S}_1 \cap \mathcal{S}_2$ (getting rid of the definition of *open distance*).

3) The dynamic consensus problem: In a consensus problem, the agents agree upon a state value by making only use of local information coming from neighboring agents. In its simplest formulation, the agents are required to converge to a state value which is a function of the initial state of the network. On the contrary, in the dynamic consensus problem, the agents are assumed to be non-autonomous and are required to converge to a state value which is a function of the time-varying reference signals given as input to the agents, most commonly the average [47], the median value [48], [49], and the maximum value [9], [50]; in [6] a unified approach has been recently proposed to solve all these consensus problems. While generalizations to the open scenarios have been carried out both for the average [41], [51]–[53] (see also [10] for an in-depth analysis of performance limitations) and the median [35] (even if only for the continuous-time case), this is not the case for the maximum value. Indeed, the open scenario has been addressed only for the standard (non dynamic) max consensus problem in [32] by gossip-based analysis.

The novelty of our research to this literature, is that of providing the Open Self-Tuning Dynamic Max-Consensus (OSTDMC) Protocol to estimate and track in a distributed way the maximum value of a set of reference signals fed to the agents, who don't need to disclose their identity or the signal they have access to (thus working in anonymous networks), and do not need to know anything about the network (such as upper bounds on its size or the diameter) or the reference signal (such as an upper bound on their derivative). The OSTDMC Protocol derives from a non-trivial extension to the open scenario of the STDMC Protocol previously formulated by us in [9], [23], whose applicability is limited by the assumption of the agents' knowledge of the inputs derivatives, not required by the newly proposed protocol. We prove stability of an OMASs executing the OSTDMC Protocol, showing its finite-time convergence and determining its stability radius, which bounds the tracking error.

B. Structure of the manuscript

Section II provides the notation and recalls the relevant concepts of graph theory applied to open networks of agents in Section II-A, the generalized definitions of state evolution, equilibria, and stability for open networks in Section II-B, and the stability criteria for the class of *paracontractive* and *slowly expansive* systems in Section II-C. Section III presents and characterizes a novel protocol called OSTDMC to solve the dynamic max-consensus problem in Open Multi-Agent systems, instrumental to the main results of this paper. Section IV presents and characterizes DR-ON-ES, an online and distributed algorithm to estimate the nodes' eccentricities and the network's size, diameter, and radius in open networks, which exploits the OSTDMC protocol. Numerical simulations are provided in Section V and concluding remarks in Section VI.

II. Background on Open Multi-Agent Systems

An open multi-agent system (OMAS) is a multi-agent system where agents can join and leave the network at any time during algorithm execution, thus it has a time-varying number of agents. We now provide some background on graph theory for OMASs.

A. Open networks and graphs

The pattern of interactions among the agents in an OMAS is modeled by a time-varying undirected graph $\mathcal{G}_k = (\mathcal{V}_k, \mathcal{E}_k)$, where $\mathcal{V}_k \subset \mathbb{N}$ is a time-varying set of nodes modeling the agents, and $\mathcal{E}_k \subseteq \mathcal{V}_k \times \mathcal{V}_k$ is a time-varying set of edges modeling the point-to-point communication channels between them. The number of agents $n_k = |\mathcal{V}_k| \in (0, \infty)$ is assumed strictly positive without any upper bound. Two agents i and j are said to be neighbors at time k if there is share a communication channel, i.e., if there is an edge $(i, j) \in \mathcal{E}_k$. Communications among the agents are assumed to be bidirectional, which implies that the graph $\mathcal{G}_k = (\mathcal{V}_k, \mathcal{E}_k)$ is undirected at any time k, i.e., for all $(i, j) \in \mathcal{E}_k$ then also $(j, i) \in \mathcal{E}_k$. The set of all undirected graphs is denoted by $\mathbb{G}$. A *path* between two nodes i and j in a graph is a sequence of consecutive and distinct edges $(i, p), (p, q), \dots, (r, s), (s, j)$ where each successive edge belongs to $\mathcal{E}_k$ and shares a distinct node with its predecessor. Among all paths between two nodes i and j, the one with the lowest number of edges is said to be the *shortest path* and it is denoted by $\pi_k^{i,j}$. An undirected graph $\mathcal{G}_k$ is said to be *connected* if there exists a path between any pair of nodes. The *distance* between two nodes $i, j \in \mathcal{V}_k$ at time k is denoted by $\text{dist}_k^{i,j}$ and it is defined as the length (number of edges) of the shortest path $\pi_k^{i,j}$, namely $\text{dist}_k^{i,j} = \left|\pi_k^{i,j}\right|$. The *eccentricity* ε_k^i of node $i \in \mathcal{V}_k$ at time k is defined as its maximal distance to any other node, $\varepsilon_k^i = \max_{j \in \mathcal{V}_k} \text{dist}_k^{i,j}$. The *diameter* δ_k of graph $\mathcal{G}_k$ at time k is defined as the maximal eccentricity among the nodes, $\delta_k = \max_{i \in \mathcal{V}_k} \varepsilon_k^i$. The *radius* ρ_k of graph $\mathcal{G}_k$ at time k is defined as the minimal eccentricity among the nodes, $\rho_k = \min_{i \in \mathcal{V}_k} \varepsilon_k^i$. The set of neighbors of the i-th agent at time k is denoted by $\mathcal{N}_{i,k} = \{j \in \mathcal{V}_k : (i, j) \in \mathcal{E}_k\}$, whose cardinality gives the number of neighbors, i.e., the degree of agent i at time k, denoted by $\eta_{i,k} = |\mathcal{N}_{i,k}|$. Note that graphs are assumed to be with self-loops, i.e., $i \in \mathcal{N}_{i,k}$. Similarly, the set of h*-hops neighbors* at time k is denoted by $\mathcal{N}_{i,k}^h$ and it comprises the set of agents j which share a path $\pi_{i,j}$ between i and j and $\text{dist}_{ij} \equiv h$. Also, $\mathcal{N}_k^h = \{i \in \mathcal{V} : \varepsilon_i(k) = h\}$ denotes the set of all nodes with eccentricity h at time k.

Due to the time-varying nature of the set-up, we identify in the set of agents $\mathcal{V}_k$ at each time k the following subsets:

- *Remaining agents* $\mathcal{R}_k = \mathcal{V}_k \cap \mathcal{V}_{k-1}$: agents present in the network at time $k-1$ and time k;
- *Arriving agents* $\mathcal{A}_k = \mathcal{V}_k \setminus \mathcal{V}_{k-1}$: agents present in the network at time k but not at time $k-1$;
- *Departing agents* $\mathcal{D}_k = \mathcal{V}_k \setminus \mathcal{V}_{k+1}$: agents present in the network at time k but not at time $k+1$.

The set of departing agents may contain both remaining and arriving agents ($\mathcal{D}_k \subset \mathcal{R}_k \cup \mathcal{A}_k$), who are instead disjoint ($\mathcal{R}_k \cap \mathcal{A}_k = \emptyset$). At time k, each agent $i \in \mathcal{V}_k$ is associated with a vector state $x_k^i \in \mathbb{R}^m$ and a vector input $u_k^i \in \mathbb{R}^p$; the vectors defined stacking these variables are denoted by x_k and u_k, respectively. Since the number of agents in the network is time-varying, the sequences $\{x_k : k \in \mathbb{N}\}$ and $\{u_k : k \in \mathbb{N}\}$ contain vectors $x_k \in \mathbb{R}^{m \cdot n_k}$, $u_k \in \mathbb{R}^{p \cdot n_k}$ whose dimension changes with k, and thus are called *open sequences*.

The state of a remaining agent $i \in \mathcal{R}_k$ is updated according to a causal evolution law $f^i : \mathbb{R}^{m \cdot n_k} \times \mathbb{R}^p \times \mathbb{G} \to \mathbb{R}^m$, while the state of an arriving agent in $i \in \mathcal{A}_k$ need to be initialized according to some rule $h^i : \mathbb{R}^p \to \mathbb{R}^m$ and the state of departing agents in $\mathcal{D}_k$ are left out from x_{k+1}, yielding:

$$x_k^i = \begin{cases} f^i(x_{k-1}, u_k^i, \mathcal{G}_{k-1}) & \text{if } i \in \mathcal{R}_k, \\ h^i(u_k^i) & \text{if } i \in \mathcal{A}_k, \end{cases} \quad k \in \mathbb{N} \setminus \{0\}, \quad (1)$$

where x_0 and $\mathcal{G}_0$ are the initial state and configuration of the network. Let us define the self-map $g_k : \mathbb{R}^{m \cdot n_k} \to \mathbb{R}^{m \cdot n_k}$ as follows

$$g_k(x) := f(x, u_k, \mathcal{G}_{k-1}), \quad (2)$$

where $f = [f^1; \cdots; f^{n_k}]$. The map g_k describes the state transition of the OMAS in the case the set of agents does not change, i.e., $x_k = g_k(x_{k-1})$ when $\mathcal{V}_k = \mathcal{V}_{k-1}$.

B. Trajectory of points of interest: existence and stability

We now introduce the concept of the *trajectory of points of interest* [41, Definition 3.1].

Definition 1 (Trajectory of points of interest). *Consider an OMAS and assume that for each $k \in \mathbb{N}$ the equation*

$$x = g_{k+1}(x).$$

has a unique solution $\hat{x}_k \in \mathbb{R}^{m \cdot n_k}$, which is called "point of interest". Then, the open sequence $\{\hat{x}_k : k \in \mathbb{N}\}$ is called the "trajectory of points of interest" (TPI) of the OMAS.

The existence of a TPI is guaranteed for some classes of OMASs: this manuscript considers the class of paracontractive OMASs, whose trajectories exhibit a contracting distance from the TPI as time progresses. Paracontractive OMASs are a superclass of contractive OMASs, where the contraction specifically refers to the distance between any two trajectories.

Definition 2 (Paracontractivity). *Let $\Gamma \geq 0$ and $T \geq 1$. An OMAS is said to be "(Γ, T)-paracontractive" w.r.t. $\|\cdot\|_\infty$ if, for each $k \geq 0$ at which holds $\mathcal{V}_k = \cdots = \mathcal{V}_{k+T-1}$, there exists $\gamma \in [0, 1)$ such that for all $x \in \mathbb{R}^{m \cdot n_k}$ it holds*

$$\|(g_{k+T} \circ \cdots \circ g_{k+1})(x) - \hat{x}_k\|_\infty \leq \max\{\gamma\|x - \hat{x}_k\|_\infty, \Gamma\}, \quad (3)$$

where $\circ$ denotes the composition operator $(f \circ h)(\cdot) = f(h(\cdot))$ and $\{\hat{x}_k : k \in \mathbb{N}\}$ is the TPI of the OMAS.

Remark 1. *The composition in eq.* (3) *is well defined under the assumption that the set of agents remain unchanged from k to $k+T-1$. Moreover, both $\hat{x}_k$ and $(g_{k+T} \circ \cdots \circ g_{k+1})(x)$ belong to $\mathbb{R}^{m \cdot n_k}$, and thus they have the same dimension, which allows to use the infinity norm to evaluate their distance.*

Since our definition of paracontractivity allows the system to be expansive at each time step, while being paracontractive over a longer time window of length T, there is the need of having a bound on the rate of expansiveness. Thus, we also introduce the definition of *slow expansiveness*.

Definition 3 (Slow expansiveness). *Let $\Lambda \geq 0$. An OMAS evolving in is said to be "Λ-slowly expansive" w.r.t. $\|\cdot\|_\infty$ if* *for all $k \geq 0$ and for all $x \in \mathbb{R}^{m \cdot n_k}$ it holds*

$$\|g_{k+1}(x) - \hat{x}_k\|_\infty \leq \|x - \hat{x}_k\|_\infty + \Lambda, \tag{4}$$

where $\{\hat{x}_k : k \in \mathbb{N}\}$ is the TPI of the OMAS.

While for autonomous (time-invariant and with no inputs) and size-invariant systems the stability is a property of an equilibrium point, in our scenario of time-varying and size-varying systems the stability becomes a property of the trajectory of points of interest, which we call *open stability*.

Definition 4 (Open stability). *Consider an OMAS with state evolution $\{x_k : k \in \mathbb{N}\}$. Its TPI $\{\hat{x}_k : k \in \mathbb{N}\}$ is said to be "open stable" w.r.t. $\|\cdot\|_\infty$ if there is a stability radius $R \geq 0$ with the following property: for every $\varepsilon > R$, there exists $\delta > 0$ such that:*

$$\|x_0 - \hat{x}_0\|_\infty < \delta \Rightarrow \|x_k - \hat{x}_k\|_\infty < \varepsilon, \quad \forall k \geq 0.$$

Open stability means that the distance between the state trajectory and the TPI remains bounded if the initial condition is chosen sufficiently close to the TPI. This bound, however, differently from standard Lyapunov stability, cannot be smaller than a minimum value R, which we call the *stability radius*. When $R = 0$, the distance between the TPI and the state trajectory can be guaranteed to remain arbitrarily small, thus implying that trajectories starting on the TPI will remain on the TPI. In open multi-agent systems, however, persistent perturbations from agents joining and leaving typically imply $R > 0$, which represents the minimal guaranteed bound on the trajectory's distance from the TPI. We now also introduce the concept of *global asymptotic open stability*.

Definition 5 (Global asymptotic open stability). *Consider an OMAS whose TPI $\{\hat{x}_k : k \in \mathbb{N}\}$ is open stable with stability radius $R \geq 0$. The TPI is said to be "globally asymptotically open stable" w.r.t. $\|\cdot\|_\infty$ if all trajectories converge to within a distance of R from the TPI:*

$$\limsup_{k\to\infty} \|x_k - \hat{x}_k\|_\infty \leq R.$$

It is important to remark that the use of the infinity norm $\|\cdot\|_\infty$ obviates the necessity for distance normalization by the number of agents. In contrast, when utilizing any other norm $\|\cdot\|_p$ with a finite $p \geq 1$, normalization becomes imperative for ensuring a fair comparison of distances evaluated in spaces of different dimensions, as highlighted in [41, Definition 3.3] for the Euclidean norm $\|\cdot\|_2$. Indeed, when the $\|\cdot\|_\infty$ is employed, the stability radius remains bounded even if the number of agents increases over time, provided that the distance between each new agent and its corresponding component in the TPI remains bounded. We provide next an illustrative example.

Example 1. *Consider the open sequence $x_k = [\mathbf{1}_k^\top, -\mathbf{1}_k^\top]^\top \in \mathbb{R}^{2k}$ generated by an OMAS – where $\mathbf{1}_k \in \mathbb{R}^k$ denotes the vector of ones with k elements – and assume that the TPI $\{\hat{x}_k : k \in \mathbb{N}\}$ is such that the vector of zeros $\hat{x}_k = \mathbf{0}_{2k} \in \mathbb{R}^{2k}$. Employing the standard Euclidean norm yields*

$$\|x_k - \hat{x}_k\|_2 = \|x_k - \mathbf{0}_{2k}\|_2 = \|x_k\|_2 = \sqrt{2k},$$

which, as $k \to \infty$, diverges even though the new components have bounded distance of 1 from the corresponding component of the TPI. This issue is naturally solved by the use of the infinity norm, which does not require normalization, indeed:

$$\|x_k - \hat{x}_k\|_\infty = \|x_k - \mathbf{0}_{2k}\|_\infty = \|x_k\|_\infty = 1.$$

C. Stability conditions for paracontractive OMASs

In order to provide sufficient conditions ensuring the stability of an OMAS, in the sense of Definition 4, it is necessary to put limits on the variation of the TPI and on the process by which the agents join and leave the OMAS during time. These limits are defined next.

Definition 6 (Bounded TPI). *The TPI $\{\hat{x}_k : k \in \mathbb{N}\}$ of an OMAS is said to have "bounded variation" if*

$$\exists B \geq 0 : \quad \max_{r\in\mathcal{R}_k} \left\|\hat{x}_k^r - \hat{x}_{k-1}^r\right\|_\infty \leq B, \quad \forall k \in \mathbb{N}.$$

Definition 7 (Bounded arrival process). *The arrival process of an OMAS with TPI $\{\hat{x}_k : k \in \mathbb{N}\}$ is said to be "bounded" if*

$$\exists H \geq 0 : \quad \max_{a\in\mathcal{A}_k} \|x_k^a - \hat{x}_k^a\|_\infty \leq H, \quad \forall k \in \mathbb{N}.$$

Definition 8 (OMAS dwell time). *The OMAS has dwell time $\Upsilon \in \mathbb{N}$ if changes in the number of agents are separated by at least Υ instants of time, i.e.,*

$$\exists \Upsilon \geq 0 : \quad \mathcal{V}_{k-1} \neq \mathcal{V}_k \Rightarrow \mathcal{V}_k = \cdots = \mathcal{V}_{k+\Upsilon}, \quad \forall k \in \mathbb{N}.$$

We conclude this section by providing a novel stability result for paracontractive OMAS whose proof can be found in our preliminary work [40, Theorem 1].

Theorem 1. *Given an OMAS, if:*

a) it is (Γ, T)-paracontractive with $\gamma \in (0, 1)$ (Def. 2);
b) it is Λ-slowly expansive (Def. 3);
c) it admits a TPI with bounded variation $B \geq 0$ (Def. 6);
d) its arrival process is bounded by $H \geq 0$ (Def. 7);
e) it has a dwell time $\Upsilon \geq T - 1$ (Def. 8).

then the TPI is globally asymptotically open stable (Def. 5) with radius

$$\begin{aligned} R &= \varrho + \min\{T-1, 1\}(\Lambda + B), \\ \varrho &= \max\left\{\frac{(T-1)\Lambda + (2T-1)B}{1-\gamma}, \Gamma + TB, H\right\}. \end{aligned} \tag{5}$$

Remark 2. *If $\Gamma = 0$ and $T = 1$, then the system is paracontractive w.r.t. $\|\cdot\|_\infty$, i.e., $\|g(x_k) - \hat{x}_k\|_\infty \leq \gamma\|x - \hat{x}_k\|_\infty$. In this case, the TPI of the OMAS is open stable with stability radius $R = \max\{B/(1-\gamma), H\}$, which is the counterpart of [41, Theorem 3.8] for the infinity norm.*

III. Dynamic Max-Consensus in Open Multi-Agent Systems

This section presents a novel protocol called *open self-tuning dynamic max-consensus* (OSTDMC), the key protocol used within the proposed distributed algorithm for estimating and tracking the graph parameters of open networks of agents, called DR-ON-ES and it is presented in the next Section IV.

Let $s_k^i \in \mathbb{R}$ be a scalar time-varying reference signal to which the i-th agent has access to. The *dynamic max-consensus problem* consists in the design of proper local update rules for estimating and tracking the maximum $\bar{s}_k \in \mathbb{R}$ among the time-varying reference signals,

$$\bar{s}_k = \max_{i \in \mathcal{V}_k} s_k^i.$$

The OSTDMC Protocol we propose requires that the agents self-tune and exchange three variables: $\xi_k^i \in \mathbb{R}$ is the main tracking variable converging nearby the objective $\bar{s}_k$; $\alpha_k^i \in \mathbb{R}$ determines the decreasing rate of the estimation variable $\xi_k^i \in \mathbb{R}$; $\mu_k^i \in \mathbb{R}$ keeps track of the maximum variation of the local reference signals s_k^i. Thus, each agent's state $x_k^i \in \mathbb{R}^3$ consists of these three variables, namely,

$$x_k^i = [x_k^{i,1}, x_k^{i,2}, x_k^{i,3}]^\top := [\xi_k^i, \mu_k^i, \alpha_k^i]^\top \in \mathbb{R}^3, \quad \forall i \in \mathcal{V}.$$

The OSTDMC Protocol is ruled by the following local updates, which makes use of two parameters $\theta \geq \beta > 0$,

$$\begin{aligned}
\xi_k^i &= \begin{cases} \max\limits_{j \in \mathcal{N}_{k-1}^i} \{\xi_{k-1}^j - \operatorname*{avg}\limits_{\ell \in \mathcal{N}_{k-1}^i} \alpha_{k-1}^\ell, s_k^i\} & \text{if } i \in \mathcal{R}_k, \\ s_k^i & \text{if } i \in \mathcal{A}_k, \end{cases} \\
\mu_k^i &= \begin{cases} \max\limits_{j \in \mathcal{N}_{k-1}^i} \{\mu_{k-1}^j, \theta + (s_k^i - s_{k-1}^i)\} & \text{if } i \in \mathcal{R}_k, \\ \beta & \text{if } i \in \mathcal{A}_k, \end{cases} \\
\alpha_k^i &= \begin{cases} \alpha_{k-1}^i & \text{if } i \in \mathcal{R}_k \wedge \xi_k^i > \xi_{k-1}^i, \\ \mu_k^i & \text{if } i \in \mathcal{R}_k \wedge \xi_k^i < \xi_{k-1}^i, \\ \beta & \text{otherwise.} \end{cases}
\end{aligned} \tag{6}$$

As the graph topology $\mathcal{G}_k = (\mathcal{V}_k, \mathcal{E}_k)$ changes dynamically, each agent $i \in \mathcal{V}_k$ in the network at time k updates its state based on the information received by its neighbors $j \in \mathcal{N}_{i,k-1}$, which may be different from the set of neighbors $\mathcal{N}_{i,k}$ at the next step, thus requiring dynamic communications and memory allocation. In the reminder of the manuscript, we refer to the local interaction rule in eq. (6) of the OSTDMC Protocol with the following notation

$$x_k^i = \mathrm{OSTDMC}_{\theta,\beta}(s_k^i, x_{k-1}^j : j \in \mathcal{N}_{i,k-1}).$$

We present the following result about the existence of the TPI of an OMAS executing the OSTDMC Protocol, along with its characterization and open stability. The proof is provided in the appendix for the sake of readability.

Theorem 2. *Consider an OMAS executing the OSTDMC Protocol under the following conditions:*

a) The graph $\mathcal{G}_k$ is undirected and connected for all $k \in \mathbb{N}$;
b) The diameter δ_k is bounded by a constant $\bar{\delta}$ for all $k \in \mathbb{N}$;
c) The OMAS has a dwell time $\Upsilon \geq \bar{\delta}$;
d) The absolute variation of the reference signals is bounded by a constant $\Pi \geq 0$, i.e.,

$$\forall i \in \mathcal{R}_k: \qquad |s_k^i - s_{k-1}^i| \leq \Pi, \qquad \forall k \geq 0. \tag{7}$$

e) The reference signals lie within a set of size $\Xi \geq 0$, i.e.,

$$\max_{i \in \mathcal{V}_k} \left|\bar{s}_k - s_k^i\right| \leq \Xi, \qquad \forall k \geq 0. \tag{8}$$

If $\theta \geq \beta > 0$, then the OMAS admits a TPI such that $\hat{\xi}_k^i \in [\bar{s}_{k+1} - \bar{\delta}\beta, \bar{s}_{k+1}]$, which is globally asymptotically open stable with radius as in eq. (5) *where*

$$\begin{aligned}
T &= \bar{\delta} + 1, & \Gamma &= (\bar{\delta} + 1)(\theta + \Pi), \\
B &= \Pi, & \Lambda &= \theta + \Pi + \bar{\delta}\beta, \\
H &= \Xi, & \gamma &= \max\{0, \tfrac{\bar{x}_0^1 - \bar{s}_1 - \beta - (\Upsilon - \bar{\delta})(\theta + \Pi)}{\left\|x_0^1 - \hat{x}_0^1\right\|_\infty}\}.
\end{aligned} \tag{9}$$

Proof: See the Appendix. ■

Theorem 2 shows that the parameter θ governs a trade-off between convergence time and tracking error in the OSTDMC Protocol. Specifically, convergence time decreases as θ increases, since the contraction rate γ is proportional to θ. Conversely, the tracking error increases with θ, as the stability radius – related to Γ – also grows with θ. In contrast, θ does not affect the steady-state error of the OSTMC Protocol, defined as the distance between the TPI and the maximum reference signal. This error is instead bounded by $\bar{\delta}\beta$ and is therefore primarily influenced by the parameter β. A practical design guideline is to select a very small value for $\beta \approx 0$ to minimize the steady-state error, and then tune $\theta \gg \beta$ to balance convergence speed and tracking accuracy effectively.

IV. Distributed Estimation of Graph Parameters in Open Multi-Agent Systems

In this section, we present and characterize a distributed algorithm enabling the dynamic tracking of diameter and radius (DR) of time-varying open networks (ON) together with nodes' eccentricities and network size (ES), which is called DR-ON-ES and its detailed implementation is given in Algorithm 1 (on the next page), with its core based on the OSTDMC Protocol introduced in the previous Section III.

Our methodology extends and improves the state-of-the-art [9], [19], [54] by dealing with networks where the number of agents may change (open networks) as well as the pattern of interactions among them (time-varying networks). The following subsections then explain in detail the strategy employed by DR-ON-ES to compute the size of the network (Section IV-A), the nodes' eccentricities (Section IV-B), and the network's diameter and radius (Section IV-C), and provide a formal characterization of the mean expected value and mean squared error.

The strategy employed by DR-ON-ES for estimating the size of the network consists in letting each node $i \in \mathcal{V}_k$ in the network pick $L \in \mathbb{N}$ random numbers $s_k^{i\ell}$ with uniform distribution in $[0, 1]$, estimate in a distributed way the maximum among them by means of the OSTDMC Protocol (on the variables $x_k^{i\ell}$), and then inferring an estimation $\tilde{n}_k^i$ of the number of agents n_k by maximum likelihood expectation. On the other hand, the strategy employed by DR-ON-ES for estimating the nodes' eccentricities and the network diameter and radius exploits the specific TPI of the OSTDMC Protocol. In particular, the TPI is such that each component i is equal to the highest $i^\star$ among all numbers diminished by a constant proportional to the distance between i and $i^\star$ in the communication graph (see eq. (25) in the proof of Theorem 2 given in the Appendix). Thus, by executing two parallel instances of the OSTDMC Protocol (on the variables $x_k^{i\ell}$, $y_k^{i\ell}$) and computing the difference between the two estimations (denoted by $\sigma_k^{i\ell}$), the agents obtain an estimate $\tilde{\varepsilon}_k^i$ of their own eccentricity ε_k^i. By running another instance of the OSTDMC Protocol (on the variables $z_k^{i\ell}$) with input signals $\sigma^{i\ell}$, the agents are able to track the maximum among all these distances for each generated number: when the highest number is attained by a node in the periphery of the network, it is a good estimate of the 1diameter; when the highest number is attained by a node in the core of the network, it is a good estimate of the radius. By taking the maximum and minimum among all these distances, the agents obtain estimates $\tilde{\delta}_k^i$, $\tilde{\rho}_k^i$ of the diameter δ_k and radius ρ_k, respectively.

We prove next the open stability of the TPI of an OMAS executing DR-ON-ES (Theorem 3) and summarize the state variables in play along for each quantity in Table I, along with the corresponding estimations.

Theorem 3. *Consider an OMAS executing DR-ON-ES and assume that:*

a) The graph $\mathcal{G}_k$ is undirected and connected for all $k \in \mathbb{N}$;
b) The diameter δ_k is bounded by a constant $\bar{\delta}$ for all $k \in \mathbb{N}$;
c) The OMAS has dwell time $\Upsilon \geq \bar{\delta}$.

If the protocol is designed with $1 > \theta \geq \beta_y > \beta_x > \beta_z > 0$, then the TPI of the OMAS is globally asymptotic open stable with $\beta = \beta_y$, $\Pi = \beta_y\theta$, and $\Xi = 1$.

Proof: DR-ON-ES consists of multiple instances of the OSTDMC Protocol. Since sufficient conditions for global asymptotic open stability of the TPI of an OMAS executing the OSTDMC Protocol have been provided in Theorem 2, the TPI of an OMAS executing DR-ON-ES is globally asymptotic open stable if the same set of conditions hold: conditions (a)-(b)-(c) of Theorem 2 hold by assumption; condition (d) of Theorem 2 holds with $\Pi = \beta_y\theta$ because the reference signals are such that $\left|s_k^{i\ell} - s_{k-1}^{i\ell}\right| = 0$ and $\left|\sigma_k^{i\ell} - \sigma_{k-1}^{i\ell}\right| \leq (\beta_y - \beta_x)\theta \leq \beta_y\theta$ for all $k \in \mathbb{N}$; condition (e) holds with $\Xi = 1$ because the reference signals satisfy $s_k^{i\ell}, \sigma_k^{i\ell} \in [0,1]$, $\forall k \in \mathbb{N}$. ∎

Algorithm 1 (DR-ON-ES): Distributed tracking of **D**iameter and **R**adius (DR) of **O**pen **N**etworks (ON) with nodes' **E**ccentricities and network **S**ize (ES)

Input: Parameters $1 > \theta \geq \beta_y > \beta_x > \beta_z > 0$, $L \in \mathbb{N}$
At each time step $k = 0, 1, 2, \dots$ **all nodes** $i \in \mathcal{V}$ **do**
if $i \in \mathcal{A}_k$ **then for** $\ell = 1, \dots, L$ **node** i **does**
**** Step 1 - Generation ****
Select numbers $s_k^{i\ell} \in [0,1]$ uniformly at random and initialize its states $x_k^{i\ell}, y_k^{i\ell}, z_k^{i\ell} \in \mathbb{R}^3$ to

$$\begin{aligned} x_k^{i\ell} &= [1, \theta, \beta_x]^\top, \quad y_k^{i\ell} = [1, \theta, \beta_y]^\top, \\ z_k^{i\ell} &= [0, \gamma\theta, \beta_z]^\top, \text{ with } \gamma := (\beta_y - \beta_x). \end{aligned} \tag{10}$$

if $i \in \mathcal{R}_k$ **then for** $\ell = 1, \dots, L$ **node** i **does**
**** Step 2 - Estimation ****
Receive $x_{k-1}^{j\ell}$, $y_{k-1}^{j\ell}$, $z_{k-1}^{j\ell}$ from $j \in \mathcal{N}_{i,k-1}$
Run two instances of the OSTDMC Protocol on $s_k^{i\ell}$ with different design parameters $\beta_y > \beta_x$

$$\begin{aligned} x_k^{i\ell} &= \text{OSTDMC}_{\theta,\beta_x}(s_k^{i\ell}, x_{k-1}^{j\ell} : j \in \mathcal{N}_{i,k-1}), \\ y_k^{i\ell} &= \text{OSTDMC}_{\theta,\beta_y}(s_k^{i\ell}, y_{k-1}^{j\ell} : j \in \mathcal{N}_{i,k-1}). \end{aligned}$$

Compute the difference between the estimations, bounding its variation by $\gamma\theta$ and saturating it in $[0,1]$

$$\sigma_k^{i\ell} = \min\{1, \sigma_{k-1}^{i\ell} + \gamma\theta, \max\{0, \sigma_{k-1}^{i\ell} - \gamma\theta, x_k^{i\ell,1} - y_k^{i\ell,1}\}\}. \tag{11}$$

Run another instance of the OSTDMC Protocol on $\sigma_k^{i\ell}$

$$z_k^{i\ell} = \text{OSTDMC}_{\gamma\theta,\beta_z}(\sigma_k^{i\ell}, z_{k-1}^{j\ell} : j \in \mathcal{N}_{i,k-1}).$$

Transmit $x_k^{j\ell}$, $y_k^{j\ell}$, $z_k^{j\ell}$ to $j \in \mathcal{N}_{i,k}$
**** Step 3 - Inference ****
Compute estimations $\tilde{\varepsilon}_k^i$ of its own eccentricty, $\tilde{n}_k^i$ of the network size, $\tilde{\delta}_k^i$, $\tilde{\rho}_k^i$ of the network diameter and radius

$$\begin{aligned} \tilde{n}_k^i &= \frac{-L}{\sum_{\ell=1}^L \ln(x_k^{i\ell,1})}, & \tilde{\delta}_k^i &= \max_{\ell=1,\dots,L} \frac{z_k^{i\ell,1}}{\gamma}, \\ \tilde{\varepsilon}_k^i &= \max_{\ell=1,\cdots,L} \frac{\sigma_k^{i\ell}}{\gamma}, & \tilde{\rho}_k^i &= \min_{\ell=1,\dots,L} \frac{z_k^{i\ell,1}}{\gamma}. \end{aligned} \tag{12}$$

A. Counting the number of agents

The methodology for counting the number of agents consists in three steps:

1) (Generation) Arriving nodes $i \in \mathcal{A}_k$ generate $L \in \mathbb{N} \setminus \{0\}$ independent random numbers and initialize their states $x_k^{i\ell} \in \mathbb{R}^3$ with $\ell \in [1, L]$ as in eq. (10);
2) (Estimation) Remaining nodes $i \in \mathcal{R}_k$ execute the OSTDMC Protocol over the variables $x_k^{i\ell}$, obtaining estimations $x_k^{i\ell,1}$ of the maximum values $\bar{s}_k^\ell = \max\{s_k^{1\ell}, s_k^{2\ell}, \dots\}$
3) (Inference) All nodes $i \in \mathcal{V}_k$ infer an estimation $\tilde{n}_k^i$ of the network's size by maximum likelihood estimation from their estimations $[x_k^{i1,1}, \dots, x_k^{iL,1}]$ as in eq. (12).

The following Theorem 4 shows that for large values of L the mean expected value approaches the real size n_k, while the mean squared error decays with the inverse of L.

Theorem 4. *Consider an OMAS executing DR-ON-ES under the assumptions of Theorem 2, and let $\hat{\tilde{n}}_k^i$ be the size estimation of the i-th agent at the current point of interest of the TPI as in Table I. Then, as $\beta_x \to 0$, it holds*

$$\mathbb{E}\left[\hat{\tilde{n}}_k^i\right] = \frac{Ln_k}{L-1}, \ \mathbb{E}\left[(\hat{\tilde{n}}_k^i - n_k)^2\right] = n_k^2 \frac{L+2}{(L-1)(L-2)},$$

where n_k is the size of the network at time k.

Proof: The likelihood function measures the fitness of a statistical model to a data sample (in our case, the maximum values $\bar{s}_{k+1}^\ell$), for given values of the unknown parameters (in our case the dimension of the network n_k). By noticing that the maximum values $\bar{s}_{k+1}^\ell$ are the n_k-th order statistics of the sets $s_{k+1}^\ell = \{s_{k+1}^{1\ell}, s_{k+1}^{2\ell}, \dots\}$ for any $\ell = 1, \dots, L$, and by noticing that they are independent and identically distributed random variables forming the sample $\{\bar{s}_{k+1}^1, \dots, \bar{s}_{k+1}^L\}$, one can compute the likelihood function, cfr. [9, Section V],

$$\mathcal{L}(n_k | s_{k+1}^\ell) = n_k^L \prod_{\ell=1}^L (\bar{s}_{k+1}^\ell)^{n_k - 1}.$$

Global quantity of interest	Quantities of interest at the i-th node			
	State variables	TPI	Estimation	Value at the TPI
Number of agents n_k	$x_k^{i\ell}$	$\hat{x}_k^{i\ell}$	$\tilde{n}_k^i = -L/\sum_{\ell=1}^{L} \ln(x_k^{i\ell,1})$	$\hat{\tilde{n}}_k^i = -L/\sum_{\ell=1}^{L} \ln(\hat{x}_k^{i\ell,1})$
Eccentricity ε_k^i	$x_k^{i\ell}, y_k^{i\ell}$	$\hat{x}_k^{i\ell},\ \hat{y}_k^{i\ell}$	$\tilde{\varepsilon}_k^i = \max_{\ell=1,\cdots,L} \dfrac{x_k^{i\ell,1} - y_k^{i\ell,1}}{\gamma}$	$\hat{\tilde{\varepsilon}}_k = \max_{\ell=1,\cdots,L} \dfrac{\hat{x}_k^{i\ell,1} - \hat{y}_k^{i\ell,1}}{\gamma}$
Network diameter δ_k	$z_k^{i\ell}$	$\hat{z}_k^{i\ell}$	$\tilde{\delta}_k^i = \max_{\ell=1,\ldots,L} \dfrac{z_k^{i\ell,1}}{\gamma}$	$\hat{\tilde{\delta}}_k^i = \max_{\ell=1,\ldots,L} \dfrac{z_k^{i\ell,1}}{\gamma}$
Network raius ρ_k	$z_k^{i\ell}$	$\hat{z}_k^{i\ell}$	$\tilde{\rho}_k^i = \min_{\ell=1,\ldots,L} \dfrac{z_k^{i\ell,1}}{\gamma}$	$\hat{\tilde{\rho}}_k^i = \min_{\ell=1,\ldots,L} \dfrac{z_k^{i\ell,1}}{\gamma}$

TABLE I

STATE VARIABLES, TPI, AND OUTPUT ESTIMATIONS OF AN OMAS EXECUTING DR-ON-ES.

The value $\tilde{n}_k$ maximizing the likelihood function is the maximum likelihood estimator of parameter n_k, given by

$$\tilde{n}_k = -L/\textstyle\sum_{\ell=1}^{L} \ln(\bar{s}_{k+1}^{\ell}). \tag{13}$$

However, the i-th agent does not have the exact knowledge of the values $\bar{s}_{k+1}^{\ell}$, but it only knows its own estimates $x_k^{i\ell,1}$, obtained by executing the OSTDMC Protocol. Thus, the best it can do is to use the values $x_k^{i\ell,1}$ instead of $\bar{s}_{k+1}^{\ell}$, yielding

$$\tilde{n}_k^i = -L/\textstyle\sum_{\ell=1}^{L} \ln(x_k^{i\ell,1}), \quad \forall i \in \mathcal{V}_k, \tag{14}$$

which is the output of DR-ON-ES. By Theorem 3, and more precisely by eq. (19) in its proof, the TPI satisfies $\hat{x}_k^{i\ell,1} \in [\bar{s}_{k+1}^{\ell} - \bar{\delta}\beta_x, \bar{s}_{k+1}^{\ell}]$ which, in the limit of $\beta_x \to 0$, yields $\hat{x}_k^{i\ell,1} = \bar{s}_{k+1}^{\ell}$. This means that, the estimation at the current point of interest of the TPI is given by

$$\hat{\tilde{n}}_k^i := -L/\textstyle\sum_{\ell=1}^{L} \ln(\hat{x}_k^{i\ell,1}) = -L/\sum_{\ell=1}^{L} \ln(\bar{s}_k^{\ell})$$

The statement of the theorem follows by noticing that such estimations $\tilde{n}_k^i$ are inverse-Gamma random variables with shape L and rate Ln_k (cfr. [9, Proof of Theorem 3]). ∎

B. Computing the node eccentricities

The methodology for computing the node eccentricities consists in three steps:

1) (Generation) Arriving nodes $i \in \mathcal{A}_k$ generate $L \in \mathbb{N} \setminus \{0\}$ independent random numbers and initialize their states $x_{k+1}^{i\ell}, y_{k+1}^{i\ell} \in \mathbb{R}^3$ with $\ell \in [1, L]$ as in eqs. (10)-(10);
2) (Estimation) Remaining nodes $i \in \mathcal{R}_k$ execute two instances of OSTDMC Protocols over the variables $x_k^{i\ell}, y_k^{i\ell}$, with different tuning parameter $\beta_y > \beta_x > 0$, obtaining two different estimations $x_k^{i\ell,1}$ and $y_k^{i\ell,1}$ of the maximum values $\bar{s}_{k+1}^{\ell} = \max\{s_{k+1}^{1\ell}, s_{k+1}^{2\ell}, \ldots\}$. Exploiting the different TPIs due to $\beta_y \neq \beta_x$, the nodes can get an estimate $\sigma_k^{i\ell}$ as in eq. (11) of their distance to the node $\ell^\star$ that generated the highest number $s_{k+1}^{i\ell^\star} = \bar{s}_{k+1}^{\ell}$;
3) (Inference) All nodes $i \in \mathcal{V}_k$ infer an estimation $\tilde{\varepsilon}_{k+1}^i$ of their eccentricity by taking the maximum among their distance to other node $[\sigma_{k+1}^{i1}, \ldots, \sigma_{k+1}^{iL}]$ as in eq. (12).

The following Theorem 5 shows that for large values of L the mean approaches the real eccentricity ε_k^i, while the mean squared error decays exponentially with L.

Theorem 5. *Consider an OMAS executing DR-ON-ES under under the assumptions of Theorem 2 and let $\hat{\tilde{\varepsilon}}_k^i$ be the eccentricity estimation of the i-th agent at the current point of interest of the TPI as in Table I. Then, it holds*

$$\mathbb{E}\left[\hat{\tilde{\varepsilon}}_k^i\right] = \varepsilon_k^i - \textstyle\sum_{\varepsilon=1}^{\varepsilon_k^i} g_k(\varepsilon, \varepsilon_k^i, L),$$
$$\mathbb{E}\left[(\hat{\tilde{\varepsilon}}_k^i - \varepsilon_k^i)^2\right] = \textstyle\sum_{\varepsilon=1}^{\varepsilon_k^i} (2\varepsilon_k^i - 2\varepsilon + 1) g_k(\varepsilon, \varepsilon_k^i, L),$$

where ε_k^i is the eccentricity of agent i at time k and the function g is defined by

$$g_k(\varepsilon, \varepsilon_k^i, L) = \left(1 - n_k^{-1} \textstyle\sum_{h=\varepsilon}^{\varepsilon_k^i} \left|\mathcal{N}_{i,k}^h\right|\right)^L. \tag{15}$$

Proof: Assuming no quantization of the real numbers, for each $\ell = 1, \ldots, L$, the maximum $\bar{s}_{k+1}^{\ell}$ among each set of numbers $s_{k+1}^{\ell} = \{s_{k+1}^{1\ell}, s_{k+1}^{2\ell}, \ldots\}$ is unique with probability one by the continuity of the distribution. Let $\ell^\star$ be the node with the highest number $s_{k+1}^{i\ell^\star} = \bar{s}_{k+1}^{\ell}$, i.e., $\ell^\star = \operatorname{argmax}_{j \in \mathcal{V}_{k+1}} s_{k+1}^{j\ell}$. By Theorem 3, and more precisely by eq. (25) in its proof, the TPI is such that $\hat{x}_k^{i\ell,1} = \bar{s}_{k+1}^{\ell^\star} - \beta_x \cdot \mathrm{dist}_k^{i,\ell^\star}$, and $\hat{y}_k^{i\ell,1} = \bar{s}_{k+1}^{\ell^\star} - \beta_y \cdot \mathrm{dist}_k^{i,\ell^\star}$. Consequently, each agent can infer its distance to $\ell^\star$ by computing the difference $\hat{x}_k^{i\ell,1} - \hat{y}_k^{i\ell,1}$ and dividing it by $(\beta_y - \beta_x)$. Therefore, agent i can infer an estimate of its eccentricity by considering the maximum distance to the nodes who selected the highest number, i.e.,

$$\tilde{\varepsilon}_k^i = \max_{\ell=1,\cdots,L} \frac{\sigma_k^{i\ell}}{\beta_y - \beta_x} = \max_{\ell=1,\cdots,L} \frac{x_k^{i\ell,1} - y_k^{i\ell,1}}{\beta_y - \beta_x},$$

which is the output of DR-ON-ES, and at the point of interest of the TPI is given by

$$\hat{\tilde{\varepsilon}}_k^i = \max\nolimits_{\ell=1,\cdots,L} \mathrm{dist}_k^{i,\ell^\star}, \tag{16}$$

We first compute the probability that the estimation $\hat{\tilde{\varepsilon}}_k^i$ is greater than or equal to $\varepsilon \in \{1, \ldots, \varepsilon_k^i\}$:

$$\mathbb{P}[\hat{\tilde{\varepsilon}}_k^i \geq \varepsilon] \overset{(i)}{=} \mathbb{P}\left[\exists \ell \in \{1, \ldots, L\} : \mathrm{dist}_k^{i,\ell^\star} \geq \varepsilon\right]$$

$$\overset{(ii)}{=} 1-\mathbb{P}\left[\forall \ell \in\{1,\dots,L\}: \operatorname{dist}_k^{i,\ell^\star} < \varepsilon\right]$$
$$\overset{(iii)}{=} 1-\left(\mathbb{P}\left[\operatorname{dist}_k^{i,\ell^\star} < \varepsilon\right]\right)^L \overset{(ii)}{=} 1-\left(1-\mathbb{P}\left[\operatorname{dist}_k^{i,\ell^\star} \geq \varepsilon\right]\right)^L$$
$$\overset{(iv)}{=} 1-\left(1-n_k^{-1}\textstyle\sum_{h=\varepsilon}^{\varepsilon_k^i}\left|\mathcal{N}_i^h\right|\right)^L \overset{(v)}{=} 1-g_k(\varepsilon,\varepsilon_k^i,L), \quad (17)$$

where (i) holds by eq. (16); (ii) hold by the complementary event; (iii) holds because all events $\left|\operatorname{dist}_k^{i,\ell^\star}\right| < \varepsilon$ for $\ell \in \{1,\dots,L\}$ are independent from each other and, in turn, the joint probability equals the product of the single events' probabilities); (iv) holds because the probability that $\ell^\star$ is such that $\operatorname{dist}_k^{i,\ell^\star} \geq \varepsilon$ can be computed by taking the ratio between number of favourite outcomes, which is the sum of all nodes with distance to node i greater or equal than ε, given by $\sum_{h=\varepsilon}^{\varepsilon_k^i} |\mathcal{N}_{i,k}^h|$, and the total number of nodes $|\mathcal{V}_k|$; (v) holds by definition of the function $g(\varepsilon,\varepsilon_k^i,L)$ as in eq. (15).

We can now derive the mean expected value as follows

$$\mathbb{E}[\hat{\tilde{\varepsilon}}_k^i] \overset{(i)}{=} \textstyle\sum_{\varepsilon=1}^{\varepsilon_k^i} \varepsilon\mathbb{P}[\hat{\tilde{\varepsilon}}_k^i = \varepsilon] = \sum_{\varepsilon=1}^{\varepsilon_k^i} \varepsilon\left(\mathbb{P}[\hat{\tilde{\varepsilon}}_k^i \geq \varepsilon] - \mathbb{P}[\hat{\tilde{\varepsilon}}_k^i \geq \varepsilon+1]\right)$$
$$= \textstyle\sum_{\varepsilon=1}^{\varepsilon_k^i} \varepsilon\mathbb{P}[\hat{\tilde{\varepsilon}}_k^i \geq \varepsilon] - \sum_{\varepsilon=2}^{\varepsilon_k^i}(\varepsilon-1)\mathbb{P}[\hat{\tilde{\varepsilon}}_k^i \geq \varepsilon] = \left(\varepsilon\mathbb{P}[\hat{\tilde{\varepsilon}}_k^i \geq \varepsilon]\right)_{\varepsilon=1}$$
$$+ \textstyle\sum_{\varepsilon=2}^{\varepsilon_k^i}\mathbb{P}[\hat{\tilde{\varepsilon}}_k^i \geq \varepsilon] = \mathbb{P}[\hat{\tilde{\varepsilon}}_k^i \geq 1] + \sum_{\varepsilon=2}^{\varepsilon_k^i}\mathbb{P}[\hat{\tilde{\varepsilon}}_k^i \geq \varepsilon] = \sum_{\varepsilon=1}^{\varepsilon_k^i}\mathbb{P}[\hat{\tilde{\varepsilon}}_k^i \geq \varepsilon]$$
$$\overset{(ii)}{=} \textstyle\sum_{\varepsilon=1}^{\varepsilon_k^i}\left(1-g(\varepsilon,\varepsilon_k^i,L)\right) = \varepsilon_k^i - \sum_{\varepsilon=1}^{\varepsilon_k^i} g(\varepsilon,\varepsilon_k^i,L), \quad (18)$$

where (i) holds because the mean expected value is calculated by multiplying each of the possible outcomes by the probability each outcome will occur and then summing all of those values; (ii) holds by eq. (17). By similar steps one can also compute the second moment, $\mathbb{E}[(\hat{\tilde{\varepsilon}}_k^i)^2] = {\varepsilon_k^i}^2 - \sum_{\varepsilon=1}^{\varepsilon_k^i}(2\varepsilon-1)g(\varepsilon,\varepsilon_k^i,L)$, and, in turn, the mean squared error:

$$\mathbb{E}\left[(\hat{\tilde{\varepsilon}}_k^i - \varepsilon_k^i)^2\right] = \overbrace{\left(\mathbb{E}[(\hat{\tilde{\varepsilon}}_k^i)^2] - \mathbb{E}^2[\hat{\tilde{\varepsilon}}_k^i]\right)}^{\text{VAR}[\hat{\tilde{\varepsilon}}_k^i]} + \overbrace{\left(\mathbb{E}[\hat{\tilde{\varepsilon}}_k^i] - \varepsilon_k^i\right)^2}^{\text{BIAS}^2[\hat{\tilde{\varepsilon}}_k^i]}$$
$$= \mathbb{E}[(\hat{\tilde{\varepsilon}}_k^i)^2] - 2\varepsilon_k^i\mathbb{E}[\hat{\tilde{\varepsilon}}_k^i] + {\varepsilon_k^i}^2 = -\textstyle\sum_{\varepsilon=1}^{\varepsilon_k^i}(2\varepsilon-1)g(\varepsilon,\varepsilon_k^i,L)$$
$$-2{\varepsilon_k^i}^2 + 2\varepsilon_k^i\textstyle\sum_{\varepsilon=1}^{\varepsilon_k^i} g(\varepsilon,\varepsilon_k^i,L) = \sum_{\varepsilon=1}^{\varepsilon_k^i}(2\varepsilon_k^i - 2\varepsilon+1)g(\varepsilon,\varepsilon_k^i,L),$$

thus completing the proof. ∎

C. Computing the network's diameter and radius

The methodology for computing the network's diameter and radius consists in three steps:

1) (Generation) Arriving nodes $i \in \mathcal{A}_k$ generate $L \in \mathbb{N}\setminus\{0\}$ independent random numbers and initialize their states $z_{k+1}^{i\ell} \in \mathbb{R}^3$ with $\ell \in [1, L]$ as in eq. (10);
2) (Estimation) Remaining nodes $i \in \mathcal{R}_k$ first estimate their distance to other nodes $\sigma_k^{i\ell}$ as in eq. (11), then execute the OSTDMC Protocol over the variables $z_k^{i\ell}$, obtaining estimations $z_k^{i\ell,1}$ of the maximum estimated distance $\bar{\sigma}_k^\ell = \max\{\sigma_k^{1\ell}, \sigma_k^{2\ell}, \dots\}$;
3) (Inference) All nodes $i \in \mathcal{V}_k$ infer estimations $\tilde{\delta}_k^i$, $\tilde{\rho}_k^i$ of the network's diameter and radius, respectively, by taking the maximum and the minimum among their estimations $[z_{k+1}^{i1,1}, \dots, z_k^{iL,1}]$ as in eq. (12);

The following Theorem 6 shows that for large values of L the mean approaches the real diameter δ_k^i or radius ρ_k^i, while the mean squared error decays exponentially with L.

Theorem 6. *Consider an OMAS executing DR-ON-ES under the assumptions of Theorem 2 and let $\hat{\tilde{\delta}}_k^i$, $\hat{\tilde{\rho}}_k^i$ be the diameter/radius estimations of the i-th agent at the current point of interest of the TPI as in Table I. Then, it holds*

$$\mathbb{E}\left[\hat{\tilde{\delta}}_k^i\right] = \delta_k - \textstyle\sum_{\delta=\rho_k+1}^{\delta_k} g_k(\delta,\delta_k,L),$$
$$\mathbb{E}\left[\hat{\tilde{\rho}}_k^i\right] = \rho_k - \textstyle\sum_{\rho=\rho_k}^{\delta_k-1} g_k(\rho,\rho_k,L),$$
$$\mathbb{E}\left[(\hat{\tilde{\delta}}_k^i - \delta_k)^2\right] = \textstyle\sum_{\delta=\rho_k+1}^{\delta_k}(2\delta_k - 2\delta+1)g_k(\delta,\delta_k L),$$
$$\mathbb{E}\left[(\hat{\tilde{\rho}}_k^i - \rho_k)^2\right] = \textstyle\sum_{\rho=\rho_k}^{\delta_k-1}(2\rho_k - 2\rho+1)g_k(\rho,\rho_k L)$$

where δ_k, ρ_k are the diameter and radius of the network at time k, the function $g_k : \mathbb{N}^3 \to \mathbb{R}_{\geq 0}$ is defined in eq. (15), and $L \in \mathbb{N}$ is the total number of randomly generated numbers.

Proof: The proof mimick that of Theorem 5 and is omitted for the sake of space. ∎

D. Discussion about the local memory requirements and the worst-case estimation errors

To execute DR-ON-ES, each node $i \in \mathcal{V}_k$ in the network at time k needs to store and update state variables $x_k^{i\ell}, y_k^{i\ell} \in \mathbb{R}^3$, $z_k^{i\ell} \in \mathbb{R}^3$ and also needs to receive and store the first two components of its neighbors' states. Consequently, each agent needs to allocate $9L+6|\mathcal{N}_{i,k-1}|L$ scalar values, i.e., the memory requirement scales as $\mathcal{O}(|\mathcal{N}_{i,k-1}|\, L)$.

A practical way to compute the maximum memory requirement for each agent in the network requires to know an upper bound on the number of neighbors (which we denote by n_{neigh}) and then choose the parameter L based on the desired estimation precision as explained next. Let n_k be the true number of agents in the network and $\hat{\tilde{n}}_k^i$ be the estimate at the TPI for the i-th agent. By the concavity of the square root and Jensen's inequality, the absolute error on the size estimation can be upper bounded by

$$\mathbb{E}\left[|\hat{\tilde{n}}_k^i - n_k|\right] \leq \sqrt{\mathbb{E}\left[(\hat{\tilde{n}}_k^i - n_k)^2\right]}.$$

Then, using Theorem 4, the percentage error satisfies

$$\varepsilon_{\%} = \frac{\mathbb{E}\left[|\hat{\tilde{n}}_k^i - n_k|\right]}{n_k} \leq \sqrt{\frac{L+2}{(L-1)(L-2)}}.$$

For instance, to guarantee a precision of $\varepsilon_{\%} = 10\%$, one should select $L \approx 100$. Assuming that the number of neighbors is at most $n_{\text{neigh}} = 10$, each agent must store approximately $9L + 6n_{\text{neigh}}L \approx 7,000$ scalars. With each scalar stored in double-precision (8 bytes per scalar), the total memory requirement is less about 50 KB. Even for $n_{\text{neigh}} = 100$, the memory requirement remains below 0.5 MB.

On the other hand, a conservative upper bound on the expected absolute error on the eccentricity, diameter, and radius estimations is discussed next. In so doing, since the estimation of the quantities is always a lower estimation and never an over estimation by construction, we can exploit the expected value of these quantities given in Theorems 5-6, which all have the same following structure:

$$\mathbb{E}\left[\left|\hat{\tilde{q}}^i-q\right|\right]=\mathbb{E}\left[q-\hat{\tilde{q}}^i\right]=\sum_{j=p}^{q}g_k(j,q,L)=n_k^{-L}\sum_{j=p}^{q}\Big(n_k-\sum_{h=j}^{q}\left|\mathcal{N}_{i,k}^h\right|\Big)^L,$$

where $q\in\{\varepsilon_k^i,\delta_k,\rho_k\}$ is the generic quantity of interest and $p\in\mathbb{N}$ is an integer. A good upper bound that works well for large small ranges $[p,q]$, can be obtained by noticing that $\sum_{h=j}^{q}|\mathcal{N}_{i,k}^h|\geq 1$, yielding

$$\sum_{j=p}^{q}g_k(j,q,L)\leq(p-q+1)\left(\frac{n_k-1}{n_k}\right)^L.$$

Instead, an upper bound that works well for large ranges $[p,q]$, can be obtained by noticing that $\sum_{h=j}^{b}\left|\mathcal{N}_{i,k}^h\right|\geq q-j+1$, which allows to write the summation as a partial Faulhaber's formula. Indeed, denoting $\kappa=n_k-q+j-1\in\{m,N\}$ with $N=n_k-1$ and $m=n_k-(q-p)+1$, yields

$$\sum_{j=p}^{q}g_k(j,q,L)\leq n_k^{-L}\sum_{j=p}^{q}\Big(n_k-q+j-1\Big)^L=:\sum_{\kappa=m}^{N}\kappa^L.$$

We find an upper bound by integral approximation:

$$\sum_{\kappa=m}^{N}\kappa^L\leq\int_m^{N+1}x^Ldx=\left[\frac{x^{L+1}}{L+1}\right]_m^{N+1}\leq\frac{(N+1)^{L+1}}{L+1}.$$

Thus, substituting yields

$$\sum_{j=p}^{q}g_k(j,q,L)\leq\frac{n_k}{(L+1)}.$$

Therefore, a good general upper bound to the expected absolute error is obtained by taking the minimum of the two above upper bounds, namely:

$$\mathbb{E}\left[\left|\hat{\tilde{\varepsilon}}_k^i-\varepsilon_k^i\right|\right]\leq\min\left\{\varepsilon_k^i\left(\frac{n_k-1}{n_k}\right)^L,\frac{n_k}{(L+1)}\right\}$$
$$\mathbb{E}\left[\left|\hat{\tilde{\delta}}_k-\delta_k\right|\right]\leq\min\left\{(\delta_k-\rho_k)\left(\frac{n_k-1}{n_k}\right)^L,\frac{n_k}{L+1}\right\},$$
$$\mathbb{E}\left[\left|\hat{\tilde{\rho}}_k-\rho_k\right|\right]\leq\min\left\{(\delta_k-\rho_k)\left(\frac{n_k-1}{n_k}\right)^L,\frac{n_k}{L+1}\right\}.$$

One can use the second upper bound, which does not depend on the quantity of interest, to determine the growth rate of L with respect to the number of n_k when one wants to achieve a maximum expected absolute error of e_{ABS}. Since these estimations are always estimation from below, due to round up operation, to achieve a expected maximum absolute error of $e_{\text{ABS}}\in\{1,2,\cdots\}$ one may just require that:

$$\frac{n_k}{(L+1)}<e_{\text{ABS}}+1\quad\Rightarrow\quad L>\frac{n}{e_{\text{ABS}}+1}-1.$$

Intuitively, to achieve a null maximum expected maximum absolute error, the nodes must select L as large as the network size, as it is usually required in flooding techniques. By allowing for larger errors, the value of L may be decreased linearly by a factor given by e_{ABS}. Another choice, could be that of having an absolute error that is proportional to the network size, namely

$$e_{\text{ABS}}=\alpha\cdot n\text{ with }\alpha\in(0,1)\qquad\Rightarrow\qquad L>\frac{1}{\alpha}.$$

In practice, the error is much less than these upper bounds, as they are computed for arbitrary graphs, including degenerate cases were these bounds are strict. In particular, such a degenerate graph is the line graph since difference between the radius and the diameter is maximized, being $\delta_k-\rho_k\approx n_k/2$ is maximized, as well as the maximum eccentricity which is exactly $\max_i\varepsilon_k^i=n_k-1$.

V. Numerical simulations

We now discuss a numerical simulation of an open network executing DR-ON-ES, which initially has $n_0=500$ nodes. At any time $k\geq 0$, there is a probability $p_k^{join}\in[0,1]$ that one node joins the network, establishing connections with any of the nodes in the network, and there is a probability $p_k^{leave}\in[0,1]$ that one node leaves the network. We model these events in a way that the network remains connected and that there exists a dwell time Υ between any two of these events. In particular, we let:

$$[p_k^{join},p_k^{leave}]=\begin{cases}[0.9,\ 0.2] & \text{if } k\leq 2\cdot 10^4\\ [0.2,\ 0.9] & \text{if } k\in(2\cdot 10^4,4\cdot 10^4]\\ [0.5,\ 0.5] & \text{if } k>4\cdot 10^4\end{cases}.$$

Consequently, for $k\leq 2\cdot 10^4$ the number the number of agent increases, for $k\in(2\cdot 10^4,4\cdot 10^4]$ decreases, then for $k>3\cdot 10^4$ oscillates around the average. The initial graph is randomly generate with diameter $\delta_0\leq 10$, and we let such value work as an upper bound on the diameter at any time, i.e., $\delta_k\leq\bar{\delta}:=10$ for all $k\geq 0$. In turn, we consider a dwell time of $\Upsilon=100\geq\bar{\delta}$. Also, we force the 1-st node to remain in the network in order to be able to show the estimation of its eccentricity over the whole simulation. The other parameters are designed as follows:

$$\theta=10^{-1},\ \beta_y=10^{-12},\ \beta_x=10^{-14},\ \beta_z=10^{-16},\ L=100.$$

Figure 1 shows the evolution of the number of agents in the network (red solid line) – experiencing a change of about 30% – and the average estimation across all agents (blue dotted line). Moreover, to better appreciate the trend of the overall estimation, it is also shown the moving average of this estimation over a period of time equal to the dwell time of the network (blue dashed line). Figure 1 reveals that DR-ON-ES enables the agents to get a reasonable estimate of the network's size by only exploiting local communications and no global information. Due to Theorem 4 and the discussion in Section IV-D, the percentage expected absolute error is bounded by $\sqrt{\frac{L+2}{(L-1)(L-2)}}\leq 10\%$, as one can verify in Figure 1. Nevertheless, the agents are able to detect the trend of the changes in the network since the execution of DR-ON-ES make the agents aware of increments or decrements of the network size.

According to Theorems 1-3, the TPI of an OMAS is globally asymptotic open stable during the execution of DR-ON-ES and its stability radius is given by $R=3.1$. Figure 2 corroborates the theoretical findings since it shows that the maximum distance from the TPI across all states $\ell = 1, \dots, L$ does not exceeds R during the whole simulation. Moreover, the fact that the moving average of such distance is exactly zero helps noticing that after every change in the network composition the network converges to the new point of interest. We remark that this does not imply that the size estimation is exact as the estimation error depends on the statistic approach employed.

Figures 3-4-5 show instead the estimation for the eccentricity of the 1-st node – who has been forced to stay within the network during the whole estimation – and the estimations for the diameter and radius. One can appreciate that in these cases the estimation errors are much lesser if compared to the size estimation error, being most of the time null and not greater than one. This is in line with Theorems 4-5-6 and the discussion in Section IV-D. Indeed, the expected absolute error is always much lesser than the theoretical upper bound $\min\left\{(\delta_k - \rho_k)\left(\frac{n_k-1}{n_k}\right)^L, \frac{n_k}{L+1}\right\} \leq 4$, as the mean squared error of the size estimation decays with the inverse of L while that of the eccentricities, diameter, and radius decays exponentially with L.

VI. Conclusions

This manuscript sets a theoretical framework to analyze stability in open systems, namely systems whose state dimension changes over time. A main novelty of this framework is that of exploiting the non-Euclidean distance based on the infinity norm, which is most suitable for open systems as it does not depend on the specific size of the state vectors. Sufficient conditions for stability are provided for paracontractive and slowly expansive system w.r.t. the infinity norm. Within this framework, it proposes, characterize, and simulate a distributed algorithm for multi-agent systems to simultaneously track the number of agents within the network, their eccentricities, and also the diameter and radius of the network. To achieve this goal, a novel protocol to solve the dynamic max-consensus in open networks.

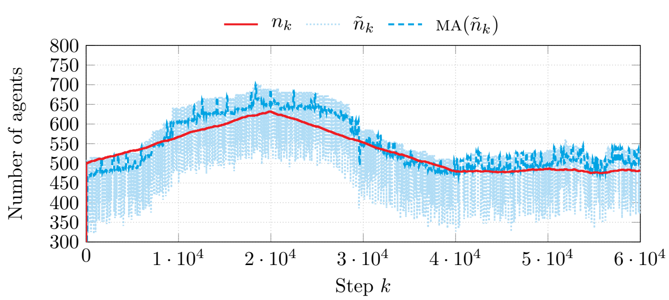

Fig. 1. Estimation of the number of agents n_k by means of DR-ON-ES. The plot shows the average $\tilde{n}_k = n_k^{-1}\sum_{i\in\mathcal{V}_k}\tilde{n}_k^i$ of the estimations across all agents and its moving average $\mathrm{MA}(\tilde{n}_k)$ on Υ steps.

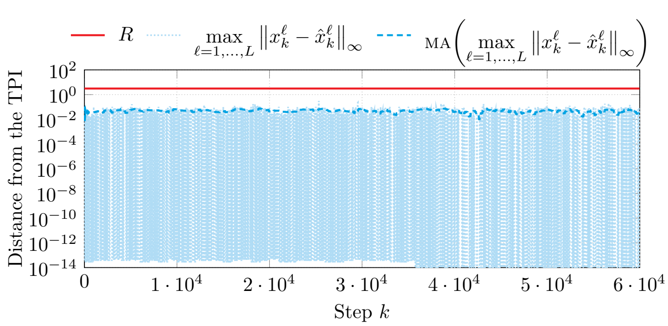

Fig. 2. Maximum distance between the network states $x_k^\ell = [x_k^{1\ell}, \cdots, x_k^{n_k\ell}]$ and the TPI $\hat{x}_k^\ell = [\hat{x}_k^{1\ell}, \cdots, \hat{x}_k^{n_k\ell}]$ across all states $\ell = 1, \dots, L$ during the execution of DR-ON-ES.

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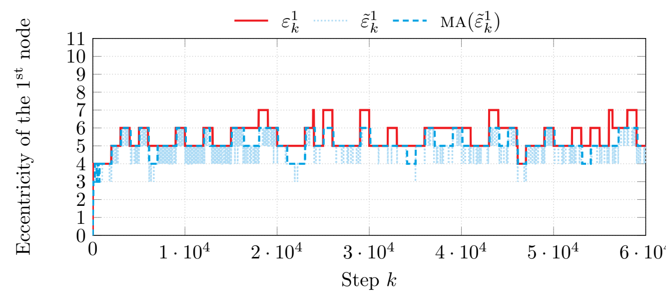

Fig. 3. Estimation of the eccentricity ε_k^1 of the 1-st node by means of DR-ON-ES. The plot shows its estimation $\tilde{\varepsilon}_k^i$ and its moving average $\mathrm{MA}(\tilde{\varepsilon}_k)$ on Υ steps.

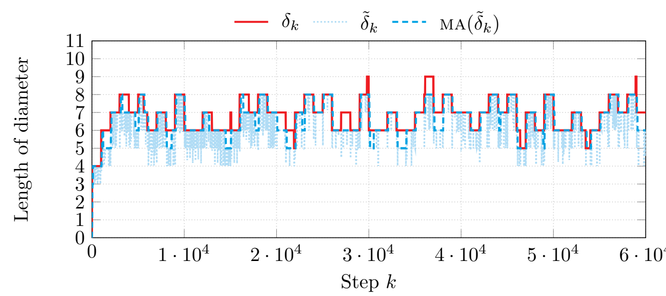

Fig. 4. Estimation of the network's diameter δ_k by means of DR-ON-ES. The plot shows the average $\tilde{\delta}_k = n_k^{-1}\sum_{i\in\mathcal{V}_k}\tilde{\delta}_k^i$ of the estimations across all agents and its moving average $\mathrm{MA}(\tilde{\delta}_k)$ on Υ steps.

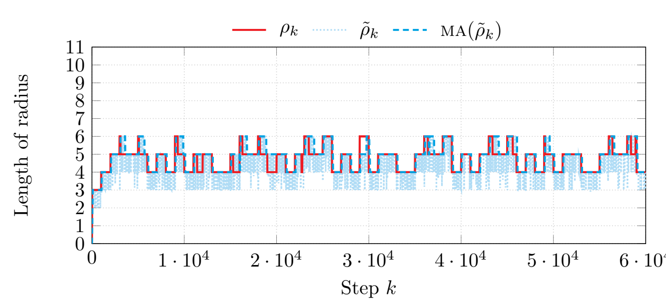

Fig. 5. Estimation of the network's radius ρ_k by means of DR-ON-ES. The plot shows the average $\tilde{\rho}_k = n_k^{-1}\sum_{i\in\mathcal{V}_k}\tilde{\rho}_k^i$ of the estimations across all agents and its moving average $\mathrm{MA}(\tilde{\rho}_k)$ on Υ steps.

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Appendix - Proof of Theorem 2

The proof requires some intermediate Lemmas, which are stated next and whose proof is given in subsequent sections.

Lemma 1. *Consider an OMAS executing the OSTDMC Protocol under the assumptions of Theorem 2. Then, the OMAS has a TPI* $\{\hat{x}_k : k \in \mathbb{N}\}$ *where* $\hat{x}_k^i = [\hat{\xi}_k^i, \mu_k^i, \alpha_k^i]^\top \in \mathbb{R}^3$ *is such that*

$$\begin{aligned} \hat{\xi}_k^i &:= \hat{x}_k^{i,1} \in [\bar{s}_{k+1} - \bar{\delta}\beta, \bar{s}_{k+1}], \\ \hat{\mu}_k^i &:= \hat{x}_k^{i,2} \in [\beta, \theta + \Pi], \qquad\qquad \forall i \in \mathcal{V}_k \\ \hat{\alpha}_k^i &:= \hat{x}_k^{i,3} = \beta. \end{aligned} \tag{19}$$

Lemma 2. *Consider an OMAS executing the OSTDMC Protocol under the assumptions of Theorem 2. If* $\theta \geq \beta > 0$*, then the OMAS is* Λ*-slowly expansive w.r.t.* $\|\cdot\|_\infty$ *where*

$$\Lambda = \theta + \Pi + \bar{\delta}\beta. \tag{20}$$

Lemma 3. *Consider an OMAS executing the OSTDMC Protocol under the assumptions of Theorem 2. If* $\theta \geq \beta > 0$*, then the OMAS is* (Γ, T)*-paracontractive w.r.t.* $\|\cdot\|_\infty$ *where*

$$T = \bar{\delta}, \qquad \Gamma = \max\{2\bar{\delta}\Pi + \bar{\delta}\theta, (\bar{\delta}+1)(\theta+\Pi)\}, \tag{21}$$

and where the contraction factor γ *for* $\Upsilon \geq T$ *is*

$$\gamma = \max\left\{0, \frac{\bar{x}_0^1 - \bar{s}_1 - \beta - (\Upsilon - \bar{\delta})(\theta + \Pi)}{\|x_0^1 - \hat{x}_0^1\|_\infty}\right\}. \tag{22}$$

We carry out the proof of Theorem 2 by showing that assumptions and conditions of Theorem 1 hold:

a) the OMAS is (Γ, T)-paracontractive w.r.t. $\|\cdot\|_\infty$ and with factor γ, where Γ and T are as in eq. (21) and γ is as in eq. (22), due to assumptions (a)-(b)-(d) and Lemma 3;

b) the OMAS is Λ-slowly expansive with Λ as in eq. (20) due to assumptions (a)-(b)-(d) and Lemma 2;

c) the OMAS admits a TPI of B-bounded variation with $B = \Pi$ due to assumptions (a)-(b)-(d) and Lemma 1;

d) the arrival process is H-bounded with $H = \Xi$ due to assumption (e).

e) the OMAS has ha dwell time Υ due to assumption (c);

Thus, Theorem 1 holds and the proof is complete.

A. Proof of Lemma 1

According to Definition 1, the TPI is the open sequence $\{\hat{x}_k : k \in \mathbb{N}\}$ where $\hat{x}_k = [\hat{\xi}_k, \hat{\mu}_k, \hat{\alpha}_k]^\top \in \mathbb{R}^{3n}$ is the unique solution to $\hat{x}_k = g_{k+1}(\hat{x}_k)$, given component-wise by

$$\begin{aligned} \hat{\xi}_k^i &= \max_{j \in \mathcal{N}_{i,k}} \{\hat{\xi}_k^j - \max_{\ell \in \mathcal{N}_{i,k}} \hat{\alpha}_k^\ell, s_{k+1}^i\} \\ \hat{\mu}_k^i &= \max_{j \in \mathcal{N}_{i,k}} \{\hat{\mu}_k^j, \theta + (s_k^i - s_{k+1}^i)\} \\ \hat{\alpha}_k^i &= \begin{cases} \hat{\alpha}_{k-1}^i & \text{if } i \in \mathcal{R}_k \wedge \hat{\xi}_k^i > \hat{\xi}_{k-1}^i, \\ \mu_k^i & \text{if } i \in \mathcal{R}_k \wedge \hat{\xi}_k^i < \hat{\xi}_{k-1}^i, \\ \beta & \text{otherwise.} \end{cases} \end{aligned} \tag{23}$$

First, variable $\hat{\mu}_k^i \geq \beta$ are initialized at $\beta \leq \theta$ and can increase up to θ plus the maximum among all state variations $s_k^i - s_{k+1}^i$, which is upper bounded by Π due to assumption (d). This proves that the component $\hat{\mu}_k^i$ is bounded as in eq. (19). Secondly, since $\hat{\xi}_k^i \neq \hat{\xi}_k^i$ is an absurd, then $\hat{\alpha}_k^i = \beta$. This proves that the component $\hat{\alpha}_k^i$ is bounded as in eq. (19).

These two preliminary results further simplify the relation of the TPI regarding the state variables ξ_k^i, as shown next

$$\hat{\xi}_k^i = \max_{j \in \mathcal{N}_{i,k}} \{\hat{\xi}_k^j - \beta, s_{k+1}^i\}. \tag{24}$$

Either one of the following case hold:

a) $\hat{\xi}_k^i = \hat{\xi}_k^j - \beta > s_{k+1}^i$ for some $j \in \mathcal{N}_{i,k} \setminus \{i\}$.

b) $\hat{\xi}_k^i = s_{k+1}^i$.

Let $\mathcal{V}_k^1$ be the set of nodes with the maximum reference signal,

$$\mathcal{V}_k^1 = \{i \in \mathcal{V}_k : s_{k+1}^i = \bar{s}_{k+1}^1\}, \text{ with } \bar{s}_{k+1}^1 := \max_{j \in \mathcal{V}_k} s_{k+1}^j.$$

By induction, for $\ell \geq 2$ let the sets of nodes with the ℓ-th maximum reference signal be

$$\mathcal{V}_k^\ell = \{i \in \mathcal{V}_k : s_{k+1}^i = \bar{s}_{k+1}^\ell\} \text{ with } \bar{s}_{k+1}^\ell := \max_{j \in \mathcal{V}_k} \{s_{k+1}^j < \bar{s}_{k+1}^{\ell-1}\}.$$

We now show by contradiction that case $a)$ cannot hold for agent $i \in \mathcal{V}_k^1$. Indeed, if $a)$ holds for some $j \in \mathcal{N}_{i,k}$, then $\hat{\xi}_k^j - \beta > s_{k+1}^i = \bar{s}_{k+1}^1 \geq s_{k+1}^j$ and, in turn, $\hat{\xi}_k^j \neq s_{k+1}^j$. Consequently, case $b)$ cannot hold for $i = j$ and, by induction, it cannot hold for all other agents. This leads to the absurd that there cannot be an agent with maximum state, i.e., for all $i \in \mathcal{V}_k$ there would be $j \in \mathcal{N}_{i,k}$ whose state is strictly greater than that of i, namely $\hat{\xi}_k^i = \hat{\xi}_k^j - \beta < \hat{\xi}_k^j$. Thus, case $a)$ leads to a contradiction and only case $b)$ can hold for $i \in \mathcal{V}_k^1$, yielding

$$\hat{\xi}_k^i = \bar{s}_{k+1}^1 \qquad \forall i \in \mathcal{V}_k^1,$$

proving the claim.

For the agents $i \in \mathcal{V}_k^2$ with the 2-nd highest reference signal, if $b)$ holds then $\hat{\xi}_k^i = \bar{s}_{k+1}^2$. If $a)$ holds instead, for

$j \in \mathcal{N}_{i,k}$ we may have two cases: if $j \in \mathcal{V}_k^1$, by the previous result for the agents with the highest reference signal, then $\hat{\xi}_k^i = \hat{\xi}_k^j - \beta = \bar{s}_{k+1}^1 - \beta$; if $\mathcal{N}_{i,k} \backslash \mathcal{V}_k^1 = \emptyset$ instead, then $j \notin \mathcal{V}_k^1$ and one must repeat the reasoning for $i = j$. This process terminates when some agent in $\mathcal{V}_k^1$ is found in a neighborhood, which is guaranteed to occur since the graph is connected and undirected according to assumption (a). The value of the component $\hat{\xi}_k^i$ takes the maximum signal $\bar{s}_{k+1}^1$ reduced by $d\beta$ where $d \in \mathbb{N}$ corresponds to the number of edges in the shortest path $\pi_k^{i,p}$ between node i and any of the nodes $p \in \mathcal{V}_k^1$:

$$\hat{\xi}_k^i = \max\{\bar{s}_{k+1}^1 - \min_{p \in \mathcal{V}_k^1} \text{dist}_k^{i,p}\beta, s_{k+1}^2\}, \qquad \forall i \in \mathcal{V}_k^2.$$

By induction, for a generic $\ell = 1, 2, \cdots$ it holds

$$\hat{\xi}_k^i = \max_{0<j<\ell}\{\bar{s}_{k+1}^j - \min_{p \in \mathcal{V}_k^{\ell-1}} \text{dist}_k^{i,p}\beta, s_{k+1}^i\}, \qquad i \in \mathcal{V}_k^\ell. \quad (25)$$

This proves that the OMAS executing the STDMC protocol has a TPI, determined component-wise by eq. (25). Since the maximum distance between two nodes at time k is equal to the diameter δ_k of the network, which is bounded by $\bar{\delta}$ by assumption (b), the components of the TPI as in eq. (25) cannot be lesser than $\bar{s}_{k+1} - \bar{\delta}\beta$, completing the proof.

B. Proof of Lemma 2

The OMAS is Λ-slowly expansive if there is $\Lambda \geq 0$ such that for any $x \in \mathbb{R}^{m \cdot n_k}$ it holds

$$\|g_{k+1}(x) - \hat{x}_k\|_\infty \leq \|x - \hat{x}_k\|_\infty + \Lambda.$$

Recalling that $x = [\cdots, x^i, \cdots]$ where each x^i has three scalar components denoted by $x^i = [\xi^i, \mu^i, \alpha^i]^\top$ let us use the use the simpler notation $x_{k+1}^i := g_{k+1}^i(x) \in \mathbb{R}^3$ and also

$$x_{k+1}^i = \begin{bmatrix} \xi_{k+1}^i, & \mu_{k+1}^i, & \alpha_{k+1}^i \end{bmatrix}^\top$$

whose specific expressions for the OSTDMC Protocol are

$$\xi_{k+1}^i = \max_{j \in \mathcal{N}_{i,k}} \{\xi^j - \operatorname*{avg}_{\ell \in \mathcal{N}_{i,k}} \alpha^\ell, s_{k+1}^i\},$$
$$\mu_{k+1}^i = \max_{j \in \mathcal{N}_{i,k}} \{\mu^j, \theta + (s_k^i - s_{k+1}^i)\},$$
$$\alpha_{k+1}^i = \begin{cases} \alpha^i & \text{if } \xi_{k+1}^i > \xi^i, \\ \mu_{k+1}^i & \text{if } \xi_{k+1}^i < \xi^i, \\ \beta & \text{otherwise.} \end{cases}$$

By assumption (d), the maximum variation of the reference signal is Π and, in turn, both μ_{k+1}^i and α_{k+1}^i are constrained within the interval $[\beta, \theta + \Pi]$, while ξ_{k+1}^i may take any value. Moreover, by Lemma 1, the TPI is such that $\hat{\mu}_k^i \in [\beta, \theta + \Pi]$ and $\hat{\alpha}_k^i = \beta$. Therefore, $\Lambda_1 = \theta + \Pi$ is an upper bound to the distances from the TPI, yielding

$$\begin{aligned} \left|\mu_{k+1}^i - \hat{\mu}_k^i\right| &\leq \Lambda_1 \leq \|\mu - \hat{\mu}_k\|_\infty + \Lambda_1, \\ \left|\alpha_{k+1}^i - \hat{\alpha}_k^i\right| &\leq \Lambda_1 \leq \|\alpha - \hat{\alpha}_k\|_\infty + \Lambda_1. \end{aligned} \quad (26)$$

Similarly, we now prove that there is $\Lambda_2 \geq 0$ such that

$$\left|\xi_{k+1}^i - \hat{\xi}_k^i\right| \leq \left\|y - \hat{\xi}_k\right\|_\infty + \Lambda_2. \quad (27)$$

By the local update rule of the OSTDMC Protocol, it holds that $\xi_{k+1}^i \in \left[\xi^i - \theta - \Pi, \max\{\bar{\xi} - \beta, \bar{s}_{k+1}\}\right]$. Moreover, by and assumption (b) and the constructing proof of Lemma 1, more precisely by eq. (25), we know that

$$\hat{\xi}_k^i \in \left[\bar{s}_{k+1} - \bar{\delta}\beta, \bar{s}_{k+1}\right], \quad (28)$$

This yields the following upper bound:

$$\begin{aligned} \left|\xi_{k+1}^i - \hat{\xi}_k^i\right| &= \max\{\xi_{k+1}^i - \hat{\xi}_k^i, \hat{\xi}_k^i - \xi_{k+1}^i\} \\ &\leq \max\left\{\begin{matrix} \max\{\bar{\xi} - \beta, \bar{s}_{k+1}\} - \bar{s}_{k+1} + \bar{\delta}\beta, \\ \bar{s}_{k+1} - \xi^i + \theta + \Pi \end{matrix}\right\} \\ &\leq \max\left\{\begin{matrix} \bar{\delta}\beta,\ \bar{\xi} - \bar{s}_{k+1} + (\bar{\delta} - 1)\beta, \\ \bar{s}_{k+1} - \xi^i + \theta + \Pi \end{matrix}\right\} \\ &\overset{(i)}{=} \max\left\{\begin{matrix} A := \bar{\delta}\beta,\ B := \bar{\xi} - \hat{\xi}_k^j + \bar{\delta}\beta,\ \forall j \in \mathcal{V}_k \\ C := \hat{\xi}_k^j - \xi^i + \theta + \Pi + \bar{\delta}\beta,\ \forall j \in \mathcal{V}_k \end{matrix}\right\}, \end{aligned}$$

where (i) holds by eq. (28) yielding $-\bar{s}_{k+1} \leq -\hat{\xi}_k^j$ and $\bar{s}_{k+1} \leq \hat{\xi}_k^j + \bar{\delta}\beta$, $\forall j \in \mathcal{V}_k$. The three possible outcomes are:

***A* $\geq$ max{*B*, *C*} is the largest -** It directly holds that

$$\left|\xi_{k+1}^i - \hat{\xi}_k^i\right| \leq \bar{\delta}\beta \leq \left\|y - \hat{\xi}_k\right\|_\infty + \underbrace{\bar{\delta}\beta}_{:=\Lambda_2'}. \quad (29)$$

***B* $\geq$ max{*A*, *C*} is the largest -** Node i has a neighbor $j^\star \in \mathcal{N}_{i,k}$ such that $\xi^{j^\star} = \bar{\xi}$, thus selecting $j = j^\star$ an upper bound can be found as follows

$$\begin{aligned} \left|\xi_{k+1}^i - \hat{\xi}_k^i\right| &\leq \xi^{j^\star} - \hat{\xi}_k^{j^\star} + \bar{\delta}\beta \overset{(i)}{=} \left|\xi^{j^\star} - \hat{\xi}_k^{j^\star} + \bar{\delta}\beta\right| \quad (30) \\ &\overset{(ii)}{\leq} \left|\xi^{j^\star} - \hat{\xi}_k^{j^\star}\right| + \bar{\delta}\beta \leq \left\|y - \hat{\xi}_k\right\|_\infty + \underbrace{\bar{\delta}\beta}_{:=\Lambda_2'}, \end{aligned}$$

where (i) holds by $B \geq A \geq 0$ and (ii) by triangle inequality.

***C* $\geq$ max{*A*, *B*} is the largest -** Node i decreases its state and thus it holds that $\xi^i \geq \xi^{j^\star}$ for any $j^\star \in \mathcal{N}_{i,k}$. Consequently, selecting $j = j^\star$ an upper bound can be found as follows

$$\begin{aligned} &\left|\xi_{k+1}^i - \hat{\xi}_k^i\right| \leq \hat{\xi}_k^{j^\star} - \xi^i + \theta + \Pi + \bar{\delta}\beta \\ &\overset{(i)}{=} \left|\hat{\xi}_k^{j^\star} - \xi^i + \theta + \Pi + \bar{\delta}\beta\right| \overset{(ii)}{\leq} \left|\hat{\xi}_k^{j^\star} - \xi^{j^\star} + \theta + \Pi + \bar{\delta}\beta\right| \\ &\overset{(iii)}{\leq} \left|\hat{\xi}_k^{j^\star} - \xi^{j^\star}\right| + \theta + \Pi + \bar{\delta}\beta \leq \left\|y - \hat{\xi}_k\right\|_\infty + \underbrace{\theta + \Pi + \bar{\delta}\beta}_{:=\Lambda_2''}, \quad (31) \end{aligned}$$

where (i) holds by assumption $C \geq A \geq 0$; (ii) holds since $\xi^i \geq \xi^{j^\star}$ for any $j^\star \in \mathcal{N}_{i,k}$ as previously noted; (iii) holds by triangle inequality.

[Conclusion] We proved that eq. (27) holds with $\Lambda_2 = \max\{\Lambda_2', \Lambda_2''\} = \Lambda_2'' = \theta + \Pi + \bar{\delta}\beta$, according to eqs. (29)-(31). The proof is completed with $\Lambda = \max\{\Lambda_1, \Lambda_2\} = \Lambda_2$,

C. Proof of Lemma 3

Let k be a generic time such that $\mathcal{V}_k = \cdots = \mathcal{V}_{k+T-1}$, then, according to Definition 2, the OMAS is (Γ, T)-paracontractive if for all $x \in \mathbb{R}^{m \cdot n_k}$ it holds

$$\left\|g_{[k,T]}(x) - \hat{x}_k\right\|_\infty \leq \max\{\gamma\|x - \hat{x}_k\|_\infty, \Gamma\}, \quad (32)$$

where $g_{[k,T]} = (g_{k+T} \circ \cdots \circ g_{k+1})$ and $\hat{x}_k$ is the TPI of the OMAS. Since $x = [\cdots, x^i, \cdots]$ where each x^i has three scalar components denoted by $x^i = [\xi^i, \mu^i, \alpha^i]^\top$, let us denote $x^i_{k+T} := g^i_{[k,T]}(x) \in \mathbb{R}^3$ and

$$x^i_{k+T} = \begin{bmatrix} \xi^i_{k+T}, & \mu^i_{k+T}, & \alpha^i_{k+T} \end{bmatrix}^\top$$

By the constructing proof of Lemma 2, and precisely from eq. (26), we know that taking $\Gamma \geq \Lambda_1 := \theta + \Pi$, ensures that eq. (32) holds for all components μ^i_{k+T} and α^i_{k+T}, i.e.,

$$\left|\mu^i_{k+T} - \hat{\mu}^i_k\right| \leq \Lambda_1, \ \left|\alpha^i_{k+T} - \hat{\alpha}^i_k\right| \leq \Lambda_1, \quad \forall i \in \mathcal{V}_k. \quad (33)$$

The rest of the proof is devoted to prove that eq. (32) holds also for the components ξ^i_{k+T}.

[Part 1: $\bar{\xi} - \beta \leq \bar{s}_{k+1}$] According to the state update rule of the OSTDMC Protocol, an agent $m \in \mathcal{V}_k$ with the maximum reference signal at time $k+1$ (there may be more than one), i.e., $s^m_{k+1} = \bar{s}_{k+1}$, updates its state to the signal itself, indeed,

$$\xi^m_{k+1} = \max_{j \in \mathcal{N}_{m,k}} \left\{\xi^j - \operatorname*{avg}_{\ell \in \mathcal{N}_{i,k}} \alpha^\ell, s^m_{k+1}\right\} = \bar{s}_{k+1},$$

because $\alpha^\ell \in [\beta, \theta + \Pi]$. Moreover, all other agents update their state to a value that is lesser than or equal to the maximum reference signal, i.e.,

$$\begin{aligned}\xi^i_{k+1} &= \max_{j \in \mathcal{N}_{i,k}} \left\{\xi^j - \operatorname*{avg}_{\ell \in \mathcal{N}_{i,k}} \alpha^\ell, s^i_{k+1}\right\} \leq \max\left\{\bar{\xi} - \beta, s^i_{k+1}\right\} \\ &\leq \max\{\bar{s}_{k+1}, s^i_{k+1}\} \leq \bar{s}_{k+1}, \qquad \forall i \in \mathcal{V}_k.\end{aligned}$$

Since the input variations are upper bounded by a constant Π due to assumption (d), and no agents join or leave the network over the time interval $[k, k+T]$, then we conclude that

$$\xi^i_{k+T} \leq \bar{s}_{k+1} + (T-1)\Pi, \qquad \forall i \in \mathcal{V}_k. \quad (34)$$

On the other hand, since $\xi^m_{k+1} = \bar{s}_{k+1}$ is the maximum among all states at time $k+1$, then all neighbors of agent m at time $k+2$ update their state according to

$$\begin{aligned}\xi^i_{k+2} &= \max_{j \in \mathcal{N}_{i,k+1}} \left\{\xi^j_{k+1} - \operatorname*{avg}_{\ell \in \mathcal{N}_{i,k}} \alpha^\ell_{k+1}, s^i_{k+2}\right\} \\ &\geq \xi^m_{k+1} - (\theta+\Pi) = \bar{s}_{k+1} - (\theta + \Pi), \quad \forall i \in \mathcal{N}_{m,k+1}.\end{aligned}$$

because $\alpha^\ell_{k+1} \in [\beta, \theta + \Pi]$. By induction, Since by (a) the network remains connected and by (b) the diameter is upper bounded by $\bar{\delta}$, then for $T \geq \bar{\delta} + 1$ all agents have updated their state such that it is greater than or equal to $\bar{s}_{k+1}$ reduced by a number of parameters $(\theta + \Pi)$ equal to T, yielding

$$\xi^i_{k+T} \geq \bar{s}_{k+1} - (T-1)(\theta + \Pi), \qquad \forall i \in \mathcal{V}_k. \quad (35)$$

Combining eqs. (34)-(35) yields

$$\xi^i_{k+T} \in [\bar{s}_{k+1} - (T-1)(\theta+\Pi), \bar{s}_{k+1} + (T-1)\Pi], \quad \forall i \in \mathcal{V}_k.$$

According to Lemma 1, the TPI satisfies $\hat{\xi}^i_k \in [\bar{s}_{k+1} - \bar{\delta}\beta, \bar{s}_{k+1}]$ and, in turn,

$$\begin{aligned}\left|\xi^i_{k+T} - \hat{\xi}^i_k\right| &\leq \max\{\bar{\xi}_{k+T} - \hat{\underline{\xi}}^i_k, \hat{\bar{\xi}}^i_k - \underline{\xi}_{k+T}\} \\ &\leq \max\{(T-1)\Pi + \bar{\delta}\beta, (T-1)(\theta+\Pi)\} \\ &\leq (T-1)(\theta+\Pi) := \Gamma_1, \quad \forall i \in \mathcal{V}_k. \quad (36)\end{aligned}$$

where $\bar{v}$, $\underline{v}$ denote the maximum and the minimum of a vector $v \in \mathbb{R}^n$, respectively. In other words, taking $\Gamma \geq \Gamma_1 := (T-1)(\theta + 2\Pi)$ ensures that eq. (32) holds for all components ξ^i_{k+T} with $i \in \mathcal{V}$.

[Part 2: $\bar{\xi} - \beta > \bar{s}_{k+1}$] According to the state update rule of the STDMC protocol, the maximum state among all agents reduces at least by a factor β, indeed, for each agent $i \in \mathcal{V}_k$ it holds that

$$\xi^i_{k+1} = \max_{j \in \mathcal{N}_{i,k}} \left\{\xi^j - \operatorname*{avg}_{\ell \in \mathcal{N}_{i,k}} \alpha^\ell, s^i_{k+1}\right\} \leq \max\{\bar{\xi} - \beta, \bar{s}_{k+1}\} = \bar{\xi} - \beta.$$

because $\alpha^\ell \in [\beta, \theta+\Pi]$. If $\bar{\xi}_{k+1} - \beta \leq \bar{s}_{k+2}$, then the proof follows from the Part 1 previously discussed. Thus, we consider the case $\bar{\xi}_{k+1} - \beta > \bar{s}_{k+2}$, which yields

$$\begin{aligned}\xi^i_{k+2} &= \max_{j \in \mathcal{N}_{i,k+1}} \left\{\xi^j_{k+1} - \operatorname*{avg}_{\ell \in \mathcal{N}_{i,k}} \alpha^\ell_{k+1}, s^i_{k+2}\right\} \\ &\leq \max\{\bar{\xi}_{k+1} - \beta, \bar{s}_{k+2}\} = \bar{\xi}_{k+1} - \beta \leq \bar{\xi} - 2\beta, \ \forall i \in \mathcal{V}_k.\end{aligned}$$

By induction, in order to not fall in the case considered in Part 1, we assume that

$$\bar{\xi}_t - \beta > \bar{s}_{t+1}, \qquad \forall t \in [k+1, k+\bar{\delta}]. \quad (37)$$

Since by (a) the network remains connected and by (b) the diameter is upper bounded by $\bar{\delta}$, then, by induction, it holds

$$\bar{\xi}_{k+\bar{\delta}+1} \leq \bar{\xi} - (\bar{\delta}+1)\beta, \qquad \forall i \in \mathcal{V}_k.$$

In this worst case scenario, after $\bar{\delta}$ steps all agents are decreased and, according to the update rule of the parameters α^i_k, they are all updated to their maximum value equal to $\theta + \Pi$. Thus, the worst case assumption in eq. (37) for $T \geq \bar{\delta}+1$ becomes $\bar{\xi}_{k+T} - (\theta+\Pi) > \bar{s}_{k+T+1}$, and, in turn,

$$\xi^i_{k+T} \leq \bar{\xi} - (\bar{\delta}+1)\beta - (T - \bar{\delta} - 1)(\theta + \Pi), \quad \forall i \in \mathcal{V}_k. \quad (38)$$

We now focus on finding a lower bound to ξ^i_{k+T+1}. First, an agent m with the maximum state at time k (there may be more than one), i.e., $\xi^m_k = \bar{\xi}_k$, updates its state according to

$$\begin{aligned}\xi^m_{k+1} &= \max_{j \in \mathcal{N}_{m,k}} \left\{\xi^j - \operatorname*{avg}_{\ell \in \mathcal{N}_{i,k}} \alpha^\ell, s^i_{k+1}\right\} \\ &\geq \max\left\{\xi^m - (\theta+\Pi), s^m_{k+1}\right\} \geq \max\left\{\bar{\xi} - (\theta+\Pi), \bar{s}_{k+1}\right\},\end{aligned}$$

because $\alpha^\ell \in [\beta, \theta+\Pi]$. Secondly, the neighbors of the agent m update their state at time $k+2$ according to

$$\begin{aligned}\xi^i_{k+2} &= \max_{j \in \mathcal{N}_{i,k}} \left\{\xi^j_{k+1} - \operatorname*{avg}_{\ell \in \mathcal{N}_{i,k}} \alpha^\ell_{k+1}, s^i_{k+2}\right\} \\ &\geq \max\left\{\xi^m_{k+1} - (\theta+\Pi), s^i_{k+2}\right\} \\ &\geq \max\left\{\xi^m_{k+1} - (\theta+\Pi), s^i_{k+1} - \Pi\right\} \\ &\geq \max\left\{\bar{\xi} - 2(\theta+\Pi), \bar{s}_{k+1} - (\theta+\Pi), s^i_{k+1} - \Pi\right\} \\ &\geq \max\left\{\bar{\xi} - 2(\theta+\Pi), \bar{s}_{k+1} - \Pi\right\}, \qquad \forall i \in \mathcal{N}_{m,k+1}.\end{aligned}$$

Since by (a) the network remains connected and by (b) the diameter is upper bounded by $\bar{\delta}$, then, by induction, for $T \geq \bar{\delta}+1$ all agents have updated their state such that

$$\xi^i_{k+T} \geq \max\left\{\bar{\xi} - T(\theta+\Pi), \bar{s}_{k+1} - (T-1)\Pi\right\}. \quad (39)$$

When $\beta < \bar{\xi} - \bar{s}_{k+1} \leq \Gamma_2$ with $T \geq \bar{\delta}+1$, eqs. (38)-(39) yield

$$\begin{aligned}\left|\xi^i_{k+T} - \hat{\xi}^i_k\right| &\leq \max\{\bar{\xi}_{k+T} - \hat{\underline{\xi}}^i_k, \hat{\bar{\xi}}^i_k - \underline{\xi}_{k+T}\} \\ &= \max\left\{\begin{matrix}\bar{\xi} - \bar{s}_{k+1} - \beta - (T - \bar{\delta} - 1)(\theta+\Pi) \\ \bar{s}_{k+1} - \bar{\xi} + T(\theta+\Pi), \ (T-1)\Pi\end{matrix}\right\}\end{aligned}$$

$$\begin{aligned} &= \max\left\{\begin{matrix}\Gamma_2-\beta-(T-\bar{\delta}-1)(\theta+\Pi)\\ -\beta+T(\theta+\Pi),\ (T-1)\Pi\end{matrix}\right\} \\ &\le T(\theta+\Pi) := \Gamma_2, \quad \forall i\in\mathcal{V}_k. \end{aligned} \tag{40}$$

It is thus clear that taking $\Gamma \ge \Gamma_2 := T(\theta+\Pi)$ ensures that eq. (32) holds for all components ξ^i_{k+T} with $i\in\mathcal{V}$ when $\beta < \bar{\xi}_k - \bar{s}_{k+1} \le \Gamma_2 \le \Gamma$. Similarly, we consider the last case when $\bar{\xi} - \bar{s}_{k+1} = \Gamma_2 + \sigma$ for $\sigma > 0$, as follows

$$\begin{aligned} \left|\xi^i_{k+T} - \hat{\xi}^i_k\right| &\le \max\{\bar{\xi}_{k+T} - \underline{\hat{\xi}}^i_k, \bar{\hat{\xi}}^i_k - \underline{\xi}_{k+T}\} \\ &\le \Gamma_2 + \sigma - \beta - (T-\bar{\delta}-1)(\theta+\Pi),\ \forall i\in\mathcal{V}_k. \end{aligned}$$

Since by construction $\left\|y - \hat{\xi}_k\right\|_\infty \ge \bar{\xi}_k - \bar{s}_{k+1} \ge \Gamma_2 + \sigma > 0$, the above relation implies that the system admits a contraction factor $\gamma_k :\in (0,1)$ for $T \ge \bar{\delta}+1$ satisfying

$$\gamma_k := \left(\bar{\xi} - \bar{s}_{k+1} - \beta - (T-\bar{\delta}-1)(\theta+\Pi)\right) / \left\|y - \hat{\xi}_k\right\|_\infty. \tag{41}$$

The worst contraction factor γ can be determined by finding the maximum among all γ_k with $k \ge 0$. By construction, it holds that $\gamma = \gamma_0 \ge \gamma_k$.

[Conclusion] Combining the result of Part 1 and Part 2 in eqs. (33)-(36)-(40)-(41), one obtains that the OMAS is (Γ, T)-paracontractive w.r.t. $\|\cdot\|_\infty$ and with contraction factor $\gamma = \gamma_0$, where the smallest value of T is $T = \bar{\delta}+1$, such that the minimum minimum value of Γ given by

$$\Gamma = \max\{\Lambda_1, \Gamma_1, \Gamma_2\} = (\bar{\delta}+1)(\theta+\Pi),$$

thus completing the proof.

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